

# Sahlqvist via Translation

Willem Conradie<sup>2</sup>, Alessandra Palmigiano<sup>\*1,2</sup>, and Zhiguang Zhao<sup>1</sup>

<sup>1</sup>Faculty of Technology, Policy and Management, Delft University of Technology, the Netherlands

<sup>2</sup>Department of Pure and Applied Mathematics, University of Johannesburg, South Africa

March 29, 2016

## Abstract

We investigate to what extent Sahlqvist-type results for nonclassical logics can be obtained by embedding into classical logic via some Gödel-type translations. We prove the correspondence theorem via translation for inductive inequalities of arbitrary signatures of normal distributive lattice expansions. We also show that canonicity-via-translation can be obtained in a similarly straightforward manner, but only in the special setting of normal bi-Heyting algebra expansions. We expand on the difficulties involved in obtaining canonicity-via-translation outside of the bi-Heyting setting.

*Keywords:* Sahlqvist theory, Gödel-Tarski translation, algorithmic correspondence, canonicity, normal distributive lattice expansions, Heyting algebras, co-Heyting algebras, bi-Heyting algebras.

*Math. Subject Class.* 03B45, 06D50, 06D10, 03G10, 06E15.

## 1 Introduction

Sahlqvist theory has a long history in normal modal logic, going back to [31]. The Sahlqvist theorem in [31] gives a syntactic definition of a class of modal formulas, the *Sahlqvist class*, each member of which defines an elementary (i.e. first-order definable) class of frames and is canonical.

Over the years, many extensions, variations and analogues of this result have appeared, including alternative proofs in e.g. [32], generalizations to arbitrary modal signatures [14], variations of the correspondence language [28, 35], Sahlqvist-type results for hybrid logics [33], various substructural logics [25, 17], mu calculus [36], enlargements of the Sahlqvist class to e.g. the *inductive* formulas of [22], to mention but a few.

A uniform and modular theory which subsumes the above results is currently emerging, and has been dubbed *unified correspondence* [4]. It is built on duality-theoretic insights [10] and uniformly exports the state-of-the-art in Sahlqvist theory from normal modal logic to a wide range of logics which include, among others, intuitionistic and distributive and general (non-distributive) lattice-based (normal modal) logics [8, 9], non-normal (regular) modal logics of arbitrary modal signature [30], hybrid logics [12], and mu-calculus [1, 2].

The breadth of this work has stimulated many and varied applications. Some are closely related to the core concerns of the theory itself, such as the understanding of the relationship between

---

<sup>\*</sup>The research of the first author has been made possible by the National Research Foundation of South Africa, Grant number 81309. The research of the second and fourth author has been made possible by the NWO Vidi grant 016.138.314, by the NWO Aspasia grant 015.008.054, and by a Delft Technology Fellowship awarded in 2013.

different methodologies for obtaining canonicity results [29, 7], or of the phenomenon of pseudocorrespondence [11]. Other, possibly surprising applications include the dual characterizations of classes of finite lattices [16], and the identification of the syntactic shape of axioms which can be translated into structural rules of a properly displayable calculus [23]. These and other results (cf. [10]) form the body of a theory called *unified correspondence* [4], a framework within which correspondence results can be formulated and proved abstracting away from specific logical signatures, and only in terms of the order-theoretic properties of the algebraic interpretations of logical connectives.

Notwithstanding the new insights and the connections with various areas of logic brought about by these developments, when limiting attention just to the Sahlqvist results for non-classical logics, a natural question to ask is whether these results could have been obtained by embedding into classical logic. Indeed, it is well known that intuitionistic logic can be interpreted into the classical modal logic S4 via the famous Gödel-McKinsey-Tarski translation [21, 27], henceforth simply the Gödel-Tarski or Gödel translation. There exist various extensions of the Gödel translation, like the one used by Wolter and Zakharyashev [38, 39] to translate intuitionistic modal logic with one  $\Box$  connective into suitable polymodal logics on a classical propositional base. Since validity is preserved and reflected under this translation, it is possible to use it to transfer many results from classical to intuitionistic modal logic. This translation has linear complexity, and would make available (an adaptation of) the SQEMA technology for Boolean polymodal logic [6] also for intuitionistic modal logic. The idea of developing Sahlqvist theory via translation has been around for a long time (besides [38, 39], see also [37, 19] and more recently, for correspondence only, [37]). In the present paper we investigate to what extent this is realizable, given the current state-of-the-art.

A hurdle that immediately presents itself is the fact that, in general, translations of the Gödel-type can run into difficulties when trying to derive correspondence results for intuitionistic modal logics, particularly if both  $\Box$  and  $\Diamond$  occur as primitive connectives. For instance, even a minimal extension of the Gödel translation to such an intuitionistic modal setting would transform the Sahlqvist inequality  $\Box\Diamond p \leq \Diamond p$  into  $\Box\Diamond\Box p \leq \Diamond\Box p$ , which is not Sahlqvist, and in fact does not even have a first-order correspondent [34]. Any translation which ‘boxes’ propositional variables would suffer from this problem (for further discussion see [4, Section 36.9]). There are more subtle translations which avoid this problem and may therefore be used as a way to fall back on the classical Sahlqvist theorem. This is done, for example, by Gehrke, Nagahashi and Venema in [19] to obtain the correspondence part of their Sahlqvist theorem for Distributive Modal Logic.

In the present paper, the question of determining the extent to which Sahlqvist correspondence and canonicity for nonclassical logics can be obtained via translation is systematically investigated in various settings, the most general of which is given by logics algebraically captured by normal distributive lattice expansions (DLEs). The starting point of our analysis is an order-theoretic reformulation of the main semantic property of the Gödel-Tarski translation. Our main conclusions are twofold. Firstly, that the *correspondence-via-translation* methodology in [19] straightforwardly generalizes to arbitrary signatures of normal distributive lattice expansions. Secondly, that the proof of *canonicity-via-translation* can be obtained in a similarly straightforward manner, but *only* in the special setting of normal *bi-Heyting algebra* expansions. As discussed in Section 6, proving canonicity *via translation* in the general normal DLE setting, if possible at all, would require techniques that are not currently available. Therefore the existing unified correspondence techniques remain the most economical route to these results.

Overall, the translation method seems inadequate to provide autonomous foundations for a general Sahlqvist theory. Besides the technical difficulties in implementing canonicity-via-translation, proceeding via translation suffers from certain inherent methodological drawbacks. In particular, any general development of Sahlqvist theory requires a uniform way to recognize Sahlqvist-type classes across logical signatures. For a treatment via translation to be significant, the specification of these

syntactic classes cannot be derived from the translation itself, but should be independent from it. The definitions of Sahlqvist and inductive inequalities provided by unified correspondence are based on the order-theoretic analysis of the connectives of the logic under consideration, and are able to provide the required independent background. Thus, the Sahlqvist via translation methodology as developed in the present paper is, in fact, yet another application of unified correspondence.

The paper is structured as follows. Section 2 collects needed preliminaries on the setting of normal DLEs, their algebraic and relational semantics, and inductive DLE-inequalities. In section 3 we provide an order-theoretic analysis of the semantic underpinnings of the Gödel-Tarski translation. Then, in Section 4, we extend the insights gained in the previous section to a class of Gödel type translations parameterized with order-types. This sets the stage for Sections 5 and 6 where we collect our results on correspondence- and canonicity-via-translation. In particular, in Section 6.2 we discuss the difficulties in extending canonicity-via-translation beyond the bi-intuitionistic setting. We conclude in Section 7.

## 2 Preliminaries

### 2.1 Language and axiomatization of basic DLE-logics

Our base language is an unspecified but fixed language  $\mathcal{L}_{\text{DLE}}$ , to be interpreted over distributive lattice expansions of compatible similarity type. We will make heavy use of the following auxiliary definition: an *order-type* over  $n \in \mathbb{N}^1$  is an  $n$ -tuple  $\varepsilon \in \{1, \partial\}^n$ . For every order type  $\varepsilon$ , we denote its *opposite* order type by  $\varepsilon^\partial$ , that is,  $\varepsilon_i^\partial = 1$  iff  $\varepsilon_i = \partial$  for every  $1 \leq i \leq n$ . For any lattice  $\mathbb{A}$ , we let  $\mathbb{A}^1 := \mathbb{A}$  and  $\mathbb{A}^\partial$  be the dual lattice, that is, the lattice associated with the converse partial order of  $\mathbb{A}$ . For any order type  $\varepsilon$ , we let  $\mathbb{A}^\varepsilon := \prod_{i=1}^n \mathbb{A}^{\varepsilon_i}$ .

The language  $\mathcal{L}_{\text{DLE}}(\mathcal{F}, \mathcal{G})$  (from now on abbreviated as  $\mathcal{L}_{\text{DLE}}$ ) takes as parameters: 1) a denumerable set  $\text{PROP}$  of proposition letters, elements of which are denoted  $p, q, r$ , possibly with indexes; 2) disjoint sets of connectives  $\mathcal{F}$  and  $\mathcal{G}$ . Each  $f \in \mathcal{F}$  and  $g \in \mathcal{G}$  has arity  $n_f \in \mathbb{N}$  (resp.  $n_g \in \mathbb{N}$ ) and is associated with some order-type  $\varepsilon_f$  over  $n_f$  (resp.  $\varepsilon_g$  over  $n_g$ ).<sup>2</sup> The terms (formulas) of  $\mathcal{L}_{\text{DLE}}$  are defined recursively as follows:

$$\varphi ::= p \mid \perp \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\overline{\varphi}) \mid g(\overline{\varphi})$$

where  $p \in \text{PROP}$ ,  $f \in \mathcal{F}$ ,  $g \in \mathcal{G}$ . Terms in  $\mathcal{L}_{\text{DLE}}$  will be denoted either by  $s, t$ , or by lowercase Greek letters such as  $\varphi, \psi, \gamma$  etc.

**Definition 1.** For any language  $\mathcal{L}_{\text{DLE}} = \mathcal{L}_{\text{DLE}}(\mathcal{F}, \mathcal{G})$ , the *basic*, or *minimal*  $\mathcal{L}_{\text{DLE}}$ -logic is a set of sequents  $\varphi \vdash \psi$ , with  $\varphi, \psi \in \mathcal{L}_{\text{DLE}}$ , which contains the following axioms:

- Sequents for lattice operations:

$$\begin{array}{llll} p \vdash p, & \perp \vdash p, & p \vdash \top, & p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r), \\ p \vdash p \vee q, & q \vdash p \vee q, & p \wedge q \vdash p, & p \wedge q \vdash q, \end{array}$$

<sup>1</sup>Throughout the paper, order-types will be typically associated with arrays of variables  $\vec{p} := (p_1, \dots, p_n)$ . When the order of the variables in  $\vec{p}$  is not specified, we will sometimes abuse notation and write  $\varepsilon(p) = 1$  or  $\varepsilon(p) = \partial$ .

<sup>2</sup>Unary  $f$  (resp.  $g$ ) will be sometimes denoted as  $\diamond$  (resp.  $\Box$ ) if the order-type is 1, and  $\triangleleft$  (resp.  $\triangleright$ ) if the order-type is  $\partial$ .

- Sequents for  $f \in \mathcal{F}$  and  $g \in \mathcal{G}$ :

$$\begin{aligned}
& f(p_1, \dots, \perp, \dots, p_{n_f}) \vdash \perp, \text{ for } \varepsilon_f(i) = 1, \\
& f(p_1, \dots, \top, \dots, p_{n_f}) \vdash \perp, \text{ for } \varepsilon_f(i) = \partial, \\
& \top \vdash g(p_1, \dots, \top, \dots, p_{n_g}), \text{ for } \varepsilon_g(i) = 1, \\
& \top \vdash g(p_1, \dots, \perp, \dots, p_{n_g}), \text{ for } \varepsilon_g(i) = \partial, \\
& f(p_1, \dots, p \vee q, \dots, p_{n_f}) \vdash f(p_1, \dots, p, \dots, p_{n_f}) \vee f(p_1, \dots, q, \dots, p_{n_f}), \text{ for } \varepsilon_f(i) = 1, \\
& f(p_1, \dots, p \wedge q, \dots, p_{n_f}) \vdash f(p_1, \dots, p, \dots, p_{n_f}) \vee f(p_1, \dots, q, \dots, p_{n_f}), \text{ for } \varepsilon_f(i) = \partial, \\
& g(p_1, \dots, p, \dots, p_{n_g}) \wedge g(p_1, \dots, q, \dots, p_{n_g}) \vdash g(p_1, \dots, p \wedge q, \dots, p_{n_g}), \text{ for } \varepsilon_g(i) = 1, \\
& g(p_1, \dots, p, \dots, p_{n_g}) \wedge g(p_1, \dots, q, \dots, p_{n_g}) \vdash g(p_1, \dots, p \vee q, \dots, p_{n_g}), \text{ for } \varepsilon_g(i) = \partial,
\end{aligned}$$

and is closed under the following inference rules:

$$\begin{array}{c}
\frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi} \quad \frac{\varphi \vdash \psi}{\varphi(\chi/p) \vdash \psi(\chi/p)} \quad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \quad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi} \\
\\
\frac{\varphi \vdash \psi}{f(p_1, \dots, \varphi, \dots, p_n) \vdash f(p_1, \dots, \psi, \dots, p_n)} (\varepsilon_f(i) = 1) \\
\\
\frac{\varphi \vdash \psi}{f(p_1, \dots, \psi, \dots, p_n) \vdash f(p_1, \dots, \varphi, \dots, p_n)} (\varepsilon_f(i) = \partial) \\
\\
\frac{\varphi \vdash \psi}{g(p_1, \dots, \varphi, \dots, p_n) \vdash g(p_1, \dots, \psi, \dots, p_n)} (\varepsilon_g(i) = 1) \\
\\
\frac{\varphi \vdash \psi}{g(p_1, \dots, \psi, \dots, p_n) \vdash g(p_1, \dots, \varphi, \dots, p_n)} (\varepsilon_g(i) = \partial).
\end{array}$$

The minimal DLE-logic is denoted by  $\mathbf{L}_{\text{DLE}}$ . For any DLE-language  $\mathcal{L}_{\text{DLE}}$ , by an *DLE-logic* we understand any axiomatic extension of the basic  $\mathcal{L}_{\text{DLE}}$ -logic in  $\mathcal{L}_{\text{DLE}}$ .

## 2.2 Algebraic and relational semantics for basic DLE-logics

The following definition captures the algebraic setting of the present paper:

**Definition 2.** For any tuple  $(\mathcal{F}, \mathcal{G})$  of disjoint sets of function symbols as above, a *distributive lattice expansion* (abbreviated as DLE) is a tuple  $\mathbb{A} = (L, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$  such that  $L$  is a bounded distributive lattice,  $\mathcal{F}^{\mathbb{A}} = \{f^{\mathbb{A}} \mid f \in \mathcal{F}\}$  and  $\mathcal{G}^{\mathbb{A}} = \{g^{\mathbb{A}} \mid g \in \mathcal{G}\}$ , such that every  $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$  (resp.  $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$ ) is an  $n_f$ -ary (resp.  $n_g$ -ary) operation on  $\mathbb{A}$ . A DLE is *normal* if every  $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$  (resp.  $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$ ) preserves finite (hence also empty) joins (resp. meets) in each coordinate with  $\varepsilon_f(i) = 1$  (resp.  $\varepsilon_g(i) = 1$ ) and reverses finite (hence also empty) meets (resp. joins) in each coordinate with  $\varepsilon_f(i) = \partial$  (resp.  $\varepsilon_g(i) = \partial$ ).<sup>3</sup> Let  $\mathbf{DLE}$  be the class of normal DLEs. Sometimes we will refer to certain DLEs as  $\mathcal{L}_{\text{DLE}}$ -algebras when we wish to emphasize that these algebras have a compatible signature with the logical language we have fixed.

<sup>3</sup> Normal DLEs are sometimes referred to as *distributive lattices with operators* (DLOs). This terminology derives from the setting of Boolean algebras with operators, in which operators are understood as operations which preserve finite (hence also empty) joins in each coordinate. Thanks to the Boolean negation, operators are typically taken as primitive connectives, and all the other operations are reduced to these. However, this terminology results somewhat ambiguous in the lattice setting, in which primitive operations are typically maps which are operators if seen as  $\mathbb{A}^\varepsilon \rightarrow \mathbb{A}^\eta$  for some order-type  $\varepsilon$  on  $n$  and some order-type  $\eta \in \{1, \partial\}$ . Rather than speaking of lattices with  $(\varepsilon, \eta)$ -operators, we then speak of normal DLEs.

In the present paper we also find it convenient to talk of (normal) Boolean algebra expansions (BAEs) (respectively, Heyting algebra expansions (HAEs), bi-Heyting algebra expansions (bHAEs)) which are structures defined as in Definition 2, but replacing the distributive lattice  $L$  with a Boolean algebra (respectively, Heyting algebra, bi-Heyting algebra).

In the remainder of the paper, we will abuse notation and write e.g.  $f$  for  $f^{\mathbb{A}}$  when this causes no confusion. Normal DLEs constitute the main semantic environment of the present paper. Henceforth, since every DLE is assumed to be normal, the adjective will be typically dropped. The class of all DLEs is equational, and can be axiomatized by the usual distributive lattice identities and the following equations for any  $f \in \mathcal{F}$  (resp.  $g \in \mathcal{G}$ ) and  $1 \leq i \leq n_f$  (resp. for each  $1 \leq j \leq n_g$ ):

- if  $\varepsilon_f(i) = 1$ , then  $f(p_1, \dots, p \vee q, \dots, p_{n_f}) = f(p_1, \dots, p, \dots, p_{n_f}) \vee f(p_1, \dots, q, \dots, p_{n_f})$ ; moreover if  $f \in \mathcal{F}_n$ , then  $f(p_1, \dots, \perp, \dots, p_{n_f}) = \perp$ ,
- if  $\varepsilon_f(i) = \partial$ , then  $f(p_1, \dots, p \wedge q, \dots, p_{n_f}) = f(p_1, \dots, p, \dots, p_{n_f}) \wedge f(p_1, \dots, q, \dots, p_{n_f})$ ; moreover if  $f \in \mathcal{F}_n$ , then  $f(p_1, \dots, \top, \dots, p_{n_f}) = \top$ ,
- if  $\varepsilon_g(j) = 1$ , then  $g(p_1, \dots, p \wedge q, \dots, p_{n_g}) = g(p_1, \dots, p, \dots, p_{n_g}) \wedge g(p_1, \dots, q, \dots, p_{n_g})$ ; moreover if  $g \in \mathcal{G}_n$ , then  $g(p_1, \dots, \top, \dots, p_{n_g}) = \top$ ,
- if  $\varepsilon_g(j) = \partial$ , then  $g(p_1, \dots, p \vee q, \dots, p_{n_g}) = g(p_1, \dots, p, \dots, p_{n_g}) \wedge g(p_1, \dots, q, \dots, p_{n_g})$ ; moreover if  $g \in \mathcal{G}_n$ , then  $g(p_1, \dots, \perp, \dots, p_{n_g}) = \top$ .

Each language  $\mathcal{L}_{\text{DLE}}$  is interpreted in the appropriate class of DLEs. In particular, for every DLE  $\mathbb{A}$ , each operation  $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$  (resp.  $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$ ) is finitely join-preserving (resp. meet-preserving) in each coordinate when regarded as a map  $f^{\mathbb{A}} : \mathbb{A}^{\varepsilon_f} \rightarrow \mathbb{A}$  (resp.  $g^{\mathbb{A}} : \mathbb{A}^{\varepsilon_g} \rightarrow \mathbb{A}$ ).

For every DLE  $\mathbb{A}$ , the symbol  $\vdash$  is interpreted as the lattice order  $\leq$ . A sequent  $\varphi \vdash \psi$  is valid in  $\mathbb{A}$  if  $h(\varphi) \leq h(\psi)$  for every homomorphism  $h$  from the  $\mathcal{L}_{\text{DLE}}$ -algebra of formulas over **PROP** to  $\mathbb{A}$ . The notation  $\mathbb{DLE} \models \varphi \vdash \psi$  indicates that  $\varphi \vdash \psi$  is valid in every DLE. Then, by means of a routine Lindenbaum-Tarski construction, it can be shown that the minimal DLE-logic  $\mathbf{L}_{\text{DLE}}$  is sound and complete with respect to its correspondent class of algebras  $\mathbb{DLE}$ , i.e. that any sequent  $\varphi \vdash \psi$  is provable in  $\mathbf{L}_{\text{DLE}}$  iff  $\mathbb{DLE} \models \varphi \vdash \psi$ .

**Definition 3.** An  $\mathcal{L}_{\text{DLE}}$ -frame is a tuple  $\mathbb{F} = (\mathbb{X}, \mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{G}})$  such that  $\mathbb{X} = (W, \leq)$  is a (nonempty) poset,  $\mathcal{R}_{\mathcal{F}} = \{R_f \mid f \in \mathcal{F}\}$ , and  $\mathcal{R}_{\mathcal{G}} = \{R_g \mid g \in \mathcal{G}\}$  such that for each  $f \in \mathcal{F}$ , the symbol  $R_f$  denotes an  $(n_f + 1)$ -ary relation on  $W$  such that for all  $\bar{w}, \bar{v} \in \mathbb{X}^{\eta_f}$ ,

$$\text{if } R_f(\bar{w}) \text{ and } \bar{w} \leq^{\eta_f} \bar{v}, \text{ then } R_f(\bar{v}), \quad (1)$$

where  $\eta_f$  is the order-type on  $n_f + 1$  defined as follows:  $\eta_f(1) = 1$  and  $\eta_f(i + 1) = \varepsilon_f^{\partial}(i)$  for each  $1 \leq i \leq n_f$ .

Likewise, for each  $g \in \mathcal{G}$ , the symbol  $R_g$  denotes an  $(n_g + 1)$ -ary relation on  $W$  such that for all  $\bar{w}, \bar{v} \in \mathbb{X}^{\eta_g}$ ,

$$\text{if } R_g(\bar{w}) \text{ and } \bar{w} \geq^{\eta_g} \bar{v}, \text{ then } R_g(\bar{v}), \quad (2)$$

where  $\eta_g$  is the order-type on  $n_g + 1$  defined as follows:  $\eta_g(1) = 1$  and  $\eta_g(i + 1) = \varepsilon_g^{\partial}(i)$  for each  $1 \leq i \leq n_g$ .

An  $\mathcal{L}_{\text{DLE}}$ -model is a tuple  $\mathbb{M} = (\mathbb{F}, V)$  such that  $\mathbb{F}$  is an  $\mathcal{L}_{\text{DLE}}$ -frame, and  $V : \text{AtProp} \rightarrow \mathcal{P}^{\uparrow}(W)$  is a persistent valuation.

The defining clauses for the interpretation of each  $f \in \mathcal{F}$  and  $g \in \mathcal{G}$  on  $\mathcal{L}_{\text{DLE}}$ -models are given as follows:

$$\begin{array}{ll}
\mathbb{M}, w \Vdash f(\vec{\varphi}) & \text{iff} \quad \text{there exists some } \vec{v} \in W^{n_f} \text{ s.t. } R_f(w, \vec{v}) \text{ and } \mathbb{M}, v_i \Vdash^{\varepsilon_f(i)} \varphi_i \text{ for each } 1 \leq i \leq n_f, \\
\mathbb{M}, w \Vdash g(\vec{\varphi}) & \text{iff} \quad \text{for any } \vec{v} \in W^{n_g}, \text{ if } R_g(w, \vec{v}) \text{ then } \mathbb{M}, v_i \Vdash^{\varepsilon_g(i)} \varphi_i \text{ for some } 1 \leq i \leq n_g,
\end{array}$$

where  $\Vdash^1$  is  $\Vdash$  and  $\Vdash^\partial$  is  $\nVdash$ .

### 2.3 Inductive $\mathcal{L}_{\text{DLE}}$ -inequalities

In the present subsection, we will report on the definition of *inductive*  $\mathcal{L}_{\text{DLE}}$ -inequalities (cf. [4, 8]).

**Definition 4 (Signed Generation Tree).** The *positive* (resp. *negative*) *generation tree* of any  $\mathcal{L}_{\text{DLE}}$ -term  $s$  is defined by labelling the root node of the generation tree of  $s$  with the sign  $+$  (resp.  $-$ ), and then propagating the labelling on each remaining node as follows:

- For any node labelled with  $\vee$  or  $\wedge$ , assign the same sign to its children nodes.
- For any node labelled with  $h \in \mathcal{F} \cup \mathcal{G}$  of arity  $n_h \geq 1$ , and for any  $1 \leq i \leq n_h$ , assign the same (resp. the opposite) sign to its  $i$ th child node if  $\varepsilon_h(i) = 1$  (resp. if  $\varepsilon_h(i) = \partial$ ).

Nodes in signed generation trees are *positive* (resp. *negative*) if are signed  $+$  (resp.  $-$ ).

Signed generation trees will be mostly used in the context of term inequalities  $s \leq t$ . In this context we will typically consider the positive generation tree  $+s$  for the left-hand side and the negative one  $-t$  for the right-hand side. We will also say that a term-inequality  $s \leq t$  is *uniform* in a given variable  $p$  if all occurrences of  $p$  in both  $+s$  and  $-t$  have the same sign, and that  $s \leq t$  is  $\varepsilon$ -*uniform* in a (sub)array  $\vec{p}$  of its variables if  $s \leq t$  is uniform in  $p$ , occurring with the sign indicated by  $\varepsilon$ , for every  $p$  in  $\vec{p}$ .

For any term  $s(p_1, \dots, p_n)$ , any order type  $\varepsilon$  over  $n$ , and any  $1 \leq i \leq n$ , an  $\varepsilon$ -*critical node* in a signed generation tree of  $s$  is a leaf node  $+p_i$  with  $\varepsilon_i = 1$  or  $-p_i$  with  $\varepsilon_i = \partial$ . An  $\varepsilon$ -*critical branch* in the tree is a branch from an  $\varepsilon$ -critical node. The intuition, which will be built upon later, is that variable occurrences corresponding to  $\varepsilon$ -critical nodes are *to be solved for*, according to  $\varepsilon$ .

For every term  $s(p_1, \dots, p_n)$  and every order type  $\varepsilon$ , we say that  $+s$  (resp.  $-s$ ) *agrees with*  $\varepsilon$ , and write  $\varepsilon(+s)$  (resp.  $\varepsilon(-s)$ ), if every leaf in the signed generation tree of  $+s$  (resp.  $-s$ ) is  $\varepsilon$ -critical. In other words,  $\varepsilon(+s)$  (resp.  $\varepsilon(-s)$ ) means that all variable occurrences corresponding to leaves of  $+s$  (resp.  $-s$ ) are to be solved for according to  $\varepsilon$ . We will also write  $+s' < *s$  (resp.  $-s' < *s$ ) to indicate that the subterm  $s'$  inherits the positive (resp. negative) sign from the signed generation tree  $*s$ . Finally, we will write  $\varepsilon(\gamma) < *s$  (resp.  $\varepsilon^\partial(\gamma) < *s$ ) to indicate that the signed subtree  $\gamma$ , with the sign inherited from  $*s$ , agrees with  $\varepsilon$  (resp. with  $\varepsilon^\partial$ ).

**Definition 5.** Nodes in signed generation trees will be called  $\Delta$ -*adjoints*, *syntactically left residual* (*SLR*), *syntactically right residual* (*SRR*), and *syntactically right adjoint* (*SRA*), according to the specification given in Table 1. A branch in a signed generation tree  $*s$ , with  $* \in \{+, -\}$ , is called a *good branch* if it is the concatenation of two paths  $P_1$  and  $P_2$ , one of which may possibly be of length 0, such that  $P_1$  is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes, and  $P_2$  consists (apart from variable nodes) only of Skeleton-nodes.

**Definition 6 (Inductive inequalities).** For any order type  $\varepsilon$  and any irreflexive and transitive relation  $<_\Omega$  on  $p_1, \dots, p_n$ , the signed generation tree  $*s$  ( $* \in \{-, +\}$ ) of a term  $s(p_1, \dots, p_n)$  is  $(\Omega, \varepsilon)$ -*inductive* if

1. for all  $1 \leq i \leq n$ , every  $\varepsilon$ -critical branch with leaf  $p_i$  is good (cf. Definition 5);

Skeleton	PIA
$\Delta$ -adjoints	SRA
$\begin{array}{c} + \quad \vee \quad \wedge \\ - \quad \wedge \quad \vee \end{array}$	$\begin{array}{c} + \quad \wedge \quad g \quad \text{with } n_g = 1 \\ - \quad \vee \quad f \quad \text{with } n_f = 1 \end{array}$
SLR	SRR
$\begin{array}{c} + \quad \wedge \quad f \quad \text{with } n_f \geq 1 \\ - \quad \vee \quad g \quad \text{with } n_g \geq 1 \end{array}$	$\begin{array}{c} + \quad \vee \quad g \quad \text{with } n_g \geq 2 \\ - \quad \wedge \quad f \quad \text{with } n_f \geq 2 \end{array}$

Table 1: Skeleton and PIA nodes for DLE.

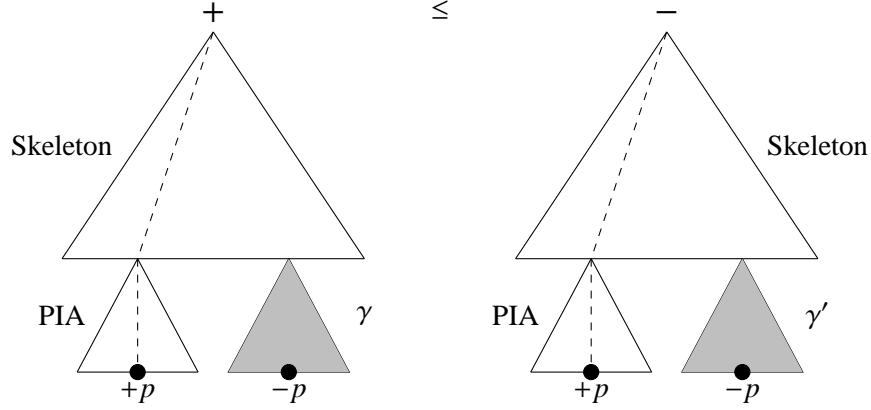


Figure 1: A schematic representation of inductive inequalities.

2. every  $m$ -ary SRR-node occurring in the critical branch is of the form  $\otimes(\gamma_1, \dots, \gamma_{j-1}, \beta, \gamma_{j+1}, \dots, \gamma_m)$ , where for any  $h \in \{1, \dots, m\} \setminus j$ :

- (a)  $\varepsilon^\partial(\gamma_h) < *s$  (cf. discussion before Definition 5), and
- (b)  $p_k <_\Omega p_i$  for every  $p_k$  occurring in  $\gamma_h$  and for every  $1 \leq k \leq n$ .

We will refer to  $<_\Omega$  as the *dependency order* on the variables. An inequality  $s \leq t$  is  $(\Omega, \varepsilon)$ -*inductive* if the signed generation trees  $+s$  and  $-t$  are  $(\Omega, \varepsilon)$ -inductive. An inequality  $s \leq t$  is *inductive* if it is  $(\Omega, \varepsilon)$ -inductive for some  $\Omega$  and  $\varepsilon$ .

### 3 The semantic environment of the Gödel-Tarski translation

In the present section, we give a semantic analysis of the Gödel-Tarski translation in an algebraic way. In what follows, for any partial order  $(W, \leq)$ , we let  $w\uparrow := \{v \in W \mid w \leq v\}$ ,  $w\downarrow := \{v \in W \mid w \geq v\}$  for every  $w \in W$ , and for every  $X \subseteq W$ , we let  $X\uparrow := \bigcup_{x \in X} x\uparrow$  and  $X\downarrow := \bigcup_{x \in X} x\downarrow$ . *Up-sets* (resp. *down-sets*) of  $(W, \leq)$  are subsets  $X \subseteq W$  such that  $X = X\uparrow$  (resp.  $X = X\downarrow$ ). We denote by  $\mathcal{P}(W)$  the Boolean algebra of subsets of  $W$ , and by  $\mathcal{P}^\uparrow(W)$  (resp.  $\mathcal{P}^\downarrow(W)$ ) the (bi-)Heyting algebra of up-sets (resp. down-sets) of  $(W, \leq)$ . Finally we let  $X^c$  denote the relative complement  $W \setminus X$  of every subset  $X \subseteq W$ .

### 3.1 Semantic analysis of the Gödel-Tarski translation

Fix a denumerable set  $\text{AtProp}$  of propositional variables. The language of intuitionistic logic over  $\text{AtProp}$  is given by

$$\mathcal{L}_I \ni \varphi ::= p \mid \perp \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi.$$

The language of the normal modal logic S4 over  $\text{AtProp}$  is given by

$$\mathcal{L}_{S4\Box} \ni \alpha ::= p \mid \perp \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg \alpha \mid \Box_{\leq} \alpha.$$

The Gödel-Tarski translation is the map  $\tau: \mathcal{L}_I \rightarrow \mathcal{L}_{S4\Box}$  defined by the following recursion:

$$\begin{aligned} \tau(p) &= \Box_{\leq} p \\ \tau(\perp) &= \perp \\ \tau(\top) &= \top \\ \tau(\varphi \wedge \psi) &= \tau(\varphi) \wedge \tau(\psi) \\ \tau(\varphi \vee \psi) &= \tau(\varphi) \vee \tau(\psi) \\ \tau(\varphi \rightarrow \psi) &= \Box_{\leq} (\neg \tau(\varphi) \vee \tau(\psi)). \end{aligned}$$

The present subsection is aimed at analyzing the semantic underpinning of the Gödel-Tarski translation. This analysis will provide the insights motivating the uniform extension of the Gödel-Tarski translation to arbitrary normal DLEs.

Both intuitionistic and S4-formulas can be interpreted on partial orders  $\mathbb{F} = (W, \leq)$ , as follows: an S4-model is a tuple  $(\mathbb{F}, U)$  where  $U: \text{AtProp} \rightarrow \mathcal{P}(W)$  is a valuation. The interpretation  $\Vdash^*$  of S4-formulas on S4-models is defined recursively as follows: for an  $w \in W$ ,

$$\begin{aligned} \mathbb{F}, w, U \Vdash^* p & \quad \text{iff } p \in U(p) \\ \mathbb{F}, w, U \Vdash^* \perp & \quad \text{never} \\ \mathbb{F}, w, U \Vdash^* \top & \quad \text{always} \\ \mathbb{F}, w, U \Vdash^* \alpha \wedge \beta & \quad \text{iff } \mathbb{F}, w, U \Vdash^* \alpha \text{ and } \mathbb{F}, w, U \Vdash^* \beta \\ \mathbb{F}, w, U \Vdash^* \alpha \vee \beta & \quad \text{iff } \mathbb{F}, w, U \Vdash^* \alpha \text{ or } \mathbb{F}, w, U \Vdash^* \beta \\ \mathbb{F}, w, U \Vdash^* \neg \alpha & \quad \text{iff } \mathbb{F}, w, U \nVdash^* \alpha \\ \mathbb{F}, w, U \Vdash^* \Box_{\leq} \alpha & \quad \text{iff } \mathbb{F}, v, U \Vdash^* \alpha \text{ for any } v \in w\uparrow. \end{aligned}$$

For any S4-formula  $\alpha$  we let  $(\llbracket \alpha \rrbracket)_U := \{w \mid \mathbb{F}, w, U \Vdash^* \alpha\}$ . It is not difficult to verify that for every  $\alpha \in \mathcal{L}_{S4}$  and any valuation  $U$ ,

$$(\llbracket \Box_{\leq} \alpha \rrbracket)_U = (\llbracket \alpha \rrbracket)_U^c \downarrow^c. \quad (3)$$

An intuitionistic model is a tuple  $(\mathbb{F}, V)$  where  $V: \text{AtProp} \rightarrow \mathcal{P}^\uparrow(W)$  is a *persistent* valuation. The interpretation  $\Vdash^*$  of S4-formulas on S4-models is defined recursively as follows: for an  $w \in W$ ,

$$\begin{aligned} \mathbb{F}, w, V \Vdash p & \quad \text{iff } p \in V(p) \\ \mathbb{F}, w, V \Vdash \perp & \quad \text{never} \\ \mathbb{F}, w, V \Vdash \top & \quad \text{always} \\ \mathbb{F}, w, V \Vdash \varphi \wedge \psi & \quad \text{iff } \mathbb{F}, w, V \Vdash \varphi \text{ and } \mathbb{F}, w, V \Vdash \psi \\ \mathbb{F}, w, V \Vdash \varphi \vee \psi & \quad \text{iff } \mathbb{F}, w, V \Vdash \varphi \text{ or } \mathbb{F}, w, V \Vdash \psi \\ \mathbb{F}, w, V \Vdash \varphi \rightarrow \psi & \quad \text{iff either } \mathbb{F}, v, V \nVdash \varphi \text{ or } \mathbb{F}, v, V \Vdash \psi \text{ for any } v \in w\uparrow. \end{aligned}$$

For any intuitionistic formula  $\varphi$  we let  $(\llbracket \varphi \rrbracket)_V := \{w \mid \mathbb{F}, w, V \Vdash \varphi\}$ . It is not difficult to verify that for all  $\varphi, \psi \in \mathcal{L}_I$  and any persistent valuation  $V$ ,

$$(\llbracket \varphi \rightarrow \psi \rrbracket)_V = (\llbracket \varphi \rrbracket)_V^c \cup (\llbracket \psi \rrbracket)_V^c \downarrow^c. \quad (4)$$

Clearly, every persistent valuation  $V$  on  $\mathbb{F}$  is also a valuation on  $\mathbb{F}$ . Moreover, for every valuation  $U$  on  $\mathcal{F}$ , the assignment mapping every  $p \in \text{AtProp}$  to  $U(p)^c \downarrow^c$  defines a persistent valuation  $U^\uparrow$  on  $\mathbb{F}$ . The main semantic property of the Gödel-Tarski translation is given by the following



**Proposition 7.** *For every intuitionistic formula  $\varphi$  and every partial order  $\mathbb{F} = (W, \leq)$ ,*

$$\mathbb{F} \Vdash \varphi \quad \text{iff} \quad \mathbb{F} \Vdash^* \tau(\varphi).$$

*Proof.* If  $\mathbb{F} \not\Vdash \varphi$ , then  $\mathbb{F}, w, V \not\Vdash \varphi$  for some persistent valuation  $V$  and  $w \in W$ . That is,  $w \notin \llbracket \varphi \rrbracket_V = \llbracket \tau(\varphi) \rrbracket_V$ , the last identity holding by item 1 of Lemma 8. Hence,  $\mathbb{F}, w, V \not\Vdash^* \tau(\varphi)$ , i.e.  $\mathbb{F} \not\Vdash^* \tau(\varphi)$ . Conversely, if  $\mathbb{F} \not\Vdash^* \tau(\varphi)$ , then  $\mathbb{F}, w, U \not\Vdash \tau(\varphi)$  for some valuation  $U$  and  $w \in W$ . That is,  $w \notin \llbracket \tau(\varphi) \rrbracket_U = \llbracket \varphi \rrbracket_{U^\uparrow}$ , the last identity holding by item 2 of Lemma 8. Hence,  $\mathbb{F}, w, U^\uparrow \not\Vdash \varphi$ , yielding  $\mathbb{F} \not\Vdash \varphi$ .  $\square$

**Lemma 8.** *For every intuitionistic formula  $\varphi$  and every partial order  $\mathbb{F} = (W, \leq)$ ,*

1.  $\llbracket \varphi \rrbracket_V = \llbracket \tau(\varphi) \rrbracket_V$  for every persistent valuation  $V$  on  $\mathbb{F}$ ;
2.  $\llbracket \tau(\varphi) \rrbracket_U = \llbracket \varphi \rrbracket_{U^\uparrow}$  for every valuation  $U$  on  $\mathbb{F}$ .

*Proof.* 1. By induction on  $\varphi$ . As for the base case, let  $\varphi := p \in \text{AtProp}$ . Then, for any persistent valuation  $V$ ,

$$\begin{aligned} \llbracket p \rrbracket_V &= V(p) && \text{(def. of } \llbracket \cdot \rrbracket_V) \\ &= V(p)^c \downarrow^c && (V \text{ persistent}) \\ &= \llbracket \Box_{\leq} p \rrbracket_V && \text{(equation (3))} \\ &= \llbracket \tau(p) \rrbracket_V, && \text{(def. of } \tau) \end{aligned}$$

as required. As for the inductive step, let  $\varphi := \psi \rightarrow \chi$ . Then, for any persistent valuation  $V$ ,

$$\begin{aligned} \llbracket \psi \rightarrow \chi \rrbracket_V &= (\llbracket \psi \rrbracket_V^c \cup \llbracket \chi \rrbracket_V)^c \downarrow^c && \text{(equation (3))} \\ &= (\llbracket \tau(\psi) \rrbracket_V^c \cup \llbracket \tau(\chi) \rrbracket_V)^c \downarrow^c && \text{(induction hypothesis)} \\ &= \llbracket \Box_{\leq} (\neg \tau(\psi) \vee \tau(\chi)) \rrbracket_V && \text{(equation (3), def. of } \llbracket \cdot \rrbracket_V) \\ &= \llbracket \tau(\psi \rightarrow \chi) \rrbracket_V, && \text{(def. of } \tau) \end{aligned}$$

as required. The remaining cases are omitted.

2. By induction on  $\varphi$ . As for the base case, let  $\varphi := p \in \text{AtProp}$ . Then, for any valuation  $U$ ,

$$\begin{aligned} \llbracket \tau(p) \rrbracket_U &= \llbracket \Box_{\leq} p \rrbracket_U && \text{(def. of } \tau) \\ &= \llbracket p \rrbracket_U^c \downarrow^c && \text{(equation (3))} \\ &= U(p)^c \downarrow^c && \text{(def. of } \llbracket \cdot \rrbracket_U) \\ &= \llbracket p \rrbracket_{U^\uparrow}, && \text{(def. of } U^\uparrow) \end{aligned}$$

as required. As for the inductive step, let  $\varphi := \psi \rightarrow \chi$ . Then, for any valuation  $U$ ,

$$\begin{aligned} \llbracket \tau(\psi \rightarrow \chi) \rrbracket_U &= \llbracket \Box_{\leq} (\neg \tau(\psi) \vee \tau(\chi)) \rrbracket_U && \text{(def. of } \tau) \\ &= \llbracket \neg \tau(\psi) \vee \tau(\chi) \rrbracket_U^c \downarrow^c && \text{(equation (3))} \\ &= (\llbracket \tau(\psi) \rrbracket_U^c \cup \llbracket \tau(\chi) \rrbracket_U)^c \downarrow^c && \text{(def. of } \llbracket \cdot \rrbracket_U) \\ &= (\llbracket \psi \rrbracket_{U^\uparrow}^c \cup \llbracket \chi \rrbracket_{U^\uparrow})^c \downarrow^c && \text{(induction hypothesis)} \\ &= \llbracket \psi \rightarrow \chi \rrbracket_{U^\uparrow}, && \text{(equation (4), } U^\uparrow \text{ persistent)} \end{aligned}$$

as required. The remaining cases are omitted.  $\square$

We saw that the key to the main semantic property of Gödel-Tarski translation, stated in Proposition 7, is the interplay between persistent and nonpersistent valuations, as captured in the above lemma. This interplay is in fact a byproduct of a more basic relationship, which we are going to analyze more in general and abstractly in the framework of interior operators.

### 3.2 An algebraic template for preservation and reflection of validity under translation

In the present subsection, we are going to generalize the key mechanism captured in the previous subsection, guaranteeing the preservation and reflection of validity under the Gödel-Tarski translation. Being able to identify this pattern in generality will make it possible to extend this mechanism to other Gödel-Tarski type translations.

Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be propositional languages over a given set  $X$ , and let  $\mathbb{A}$  and  $\mathbb{B}$  be ordered  $\mathcal{L}_1$ - and  $\mathcal{L}_2$ -algebras respectively, such that an order-embedding  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  exists. For each  $V \in \mathbb{A}^X$  and  $U \in \mathbb{B}^X$ , let  $\llbracket \cdot \rrbracket_V$  and  $\llbracket \cdot \rrbracket_U$  denote their unique homomorphic extensions to  $\mathcal{L}_1$  and  $\mathcal{L}_2$  respectively. Clearly,  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  lifts to a map  $\bar{e}: \mathbb{A}^X \rightarrow \mathbb{B}^X$  by the assignment  $V \mapsto e \circ V$ .

**Proposition 9.** *Let  $\tau: \mathcal{L}_1 \rightarrow \mathcal{L}_2$  and  $r: \mathbb{B}^X \rightarrow \mathbb{A}^X$  be such that the following conditions hold for every  $\varphi \in \mathcal{L}_1$ :*

- (a)  $e(\llbracket \varphi \rrbracket_V) = \llbracket \tau(\varphi) \rrbracket_{\bar{e}(V)}$  for every  $V \in \mathbb{A}^X$ ;
- (b)  $\llbracket \tau(\varphi) \rrbracket_U = e(\llbracket \varphi \rrbracket_{r(U)})$  for every  $U \in \mathbb{B}^X$ .

Then, for all  $\varphi, \psi \in \mathcal{L}_1$ ,

$$\mathbb{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbb{B} \models \tau(\varphi) \leq \tau(\psi).$$

*Proof.* From left to right, suppose contrapositively that  $(\mathbb{B}, U) \not\models \tau(\varphi) \leq \tau(\psi)$  for some  $U \in \mathbb{B}^X$ , that is,  $\llbracket \tau(\varphi) \rrbracket_U \not\leq \llbracket \tau(\psi) \rrbracket_U$ . By item (b) above, this non-inequality is equivalent to  $e(\llbracket \varphi \rrbracket_{r(U)}) \not\leq e(\llbracket \psi \rrbracket_{r(U)})$ , which, by the monotonicity of  $e$ , implies that  $\llbracket \varphi \rrbracket_{r(U)} \not\leq \llbracket \psi \rrbracket_{r(U)}$ , that is,  $(\mathbb{A}, r(U)) \not\models \varphi \leq \psi$ , as required. Conversely, if  $(\mathbb{A}, V) \not\models \varphi \leq \psi$  for some  $V \in \mathbb{A}^X$ , then  $\llbracket \varphi \rrbracket_V \not\leq \llbracket \psi \rrbracket_V$ , and hence, since  $e$  is an order-embedding and by item (a) above,  $\llbracket \tau(\varphi) \rrbracket_{\bar{e}(V)} = e(\llbracket \varphi \rrbracket_V) \not\leq e(\llbracket \psi \rrbracket_V) = \llbracket \tau(\psi) \rrbracket_{\bar{e}(V)}$ , that is  $(\mathbb{B}, \bar{e}(V)) \not\models \tau(\varphi) \leq \tau(\psi)$ , as required.  $\square$

Notice that in the proof above we have only made use of the assumption that  $e$  is an order-embedding, but have not needed to assume any property of  $r$ . Notice also that the proposition above is independent of the logical/algebraic signature of choice, and holds for *arbitrary* algebras. This latter point will be key to the treatment of Sahlqvist canonicity via translation.

### 3.3 Interior operator analysis of the Gödel-Tarski translation

As observed above, Proposition 9 generalizes Proposition 7 in more than one way. In the present subsection, we show that the Gödel-Tarski translation fits the strengthening given by Proposition 9. Towards this goal, we let  $X := \text{AtProp}$ ,  $\mathcal{L}_1 := \mathcal{L}_I$ , and  $\mathcal{L}_2 := \mathcal{L}_{S4}$ . Moreover, we let  $\mathbb{A}$  be a Heyting algebra, and  $\mathbb{B}$  a Boolean algebra such that an order-embedding  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  exists, which is also a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$ , and has a right adjoint<sup>4</sup>  $\iota: \mathbb{B} \rightarrow \mathbb{A}$  such that for all  $a, b \in \mathbb{A}$ ,

$$a \rightarrow^{\mathbb{A}} b = \iota(\neg^{\mathbb{B}} e(a) \vee^{\mathbb{B}} e(b)). \quad (5)$$

Then  $\mathbb{B}$  can be endowed with a natural structure of Boolean algebra expansion (BAE) by defining  $\Box^{\mathbb{B}}: \mathbb{B} \rightarrow \mathbb{B}$  by the assignment  $b \mapsto (e \circ \iota)(b)$ . The following is a well known fact in algebraic modal logic:

---

<sup>4</sup>That is,  $e(a) \leq b$  iff  $a \leq \iota(b)$  for every  $a \in \mathbb{A}$  and  $b \in \mathbb{B}$ . By well known order-theoretic facts (cf. [13]),  $e \circ \iota$  is an *interior operator*, that is, for every  $b, b' \in \mathbb{B}$ ,

- i1.  $(e \circ \iota)(b) \leq b$ ;
- i2. if  $b \leq b'$  then  $(e \circ \iota)(b) \leq (e \circ \iota)(b')$ ;
- i3.  $(e \circ \iota)(b) \leq (e \circ \iota)((e \circ \iota)(b))$ .

Moreover,  $e \circ \iota \circ e = e$  and  $\iota = \iota \circ e \circ \iota$  (cf. [13, Lemma 7.26]).

**Proposition 10.** *The BAE  $(\mathbb{B}, \Box^{\mathbb{B}})$ , with  $\Box^{\mathbb{B}}$  defined above, is normal and is also an S4-modal algebra.*

*Proof.* The fact that  $\Box^{\mathbb{B}}$  preserves finite (hence empty) meets readily follows from the fact that  $\iota$  is a right adjoint, and hence preserves existing (thus all finite) meets of  $\mathbb{B}$ , and  $e$  is a lattice homomorphism. For every  $b \in \mathbb{B}$ ,  $\iota(b) \leq \iota(b)$  implies that  $\Box^{\mathbb{B}}b = e(\iota(b)) \leq b$ , which proves (T). For every  $b \in \mathbb{B}$ ,  $e(\iota(b)) \leq e(\iota(b))$  implies that  $\iota(b) \leq \iota(e(\iota(b)))$  and hence  $\Box^{\mathbb{B}}b = e(\iota(b)) \leq e(\iota(e(\iota(b)))) = (e \circ \iota)((e \circ \iota)(b)) = \Box^{\mathbb{B}}\Box^{\mathbb{B}}b$ , which proves K4.  $\square$

Finally, we let  $r : \mathbb{B}^X \rightarrow \mathbb{A}^X$  be defined by the assignment  $U \mapsto (\iota \circ U)$ .

**Proposition 11.** *Let  $\mathbb{A}$ ,  $\mathbb{B}$ ,  $e : \mathbb{A} \hookrightarrow \mathbb{B}$  and  $r : \mathbb{B}^X \rightarrow \mathbb{A}^X$  be as above.<sup>5</sup> Then the Gödel-Tarski translation  $\tau$  satisfies conditions (a) and (b) of Proposition 9 for any formula  $\varphi \in \mathcal{L}_I$ .*

*Proof.* By induction on  $\varphi$ . As for the base case, let  $\varphi := p \in \text{AtProp}$ . Then, for any  $U \in \mathbb{B}^X$  and  $V \in \mathbb{A}^X$ ,

$$\begin{aligned}
 e(\llbracket p \rrbracket_{r(U)}) &= e((\iota \circ U)(p)) & \llbracket \tau(p) \rrbracket_{\bar{e}(V)} &= \llbracket \Box_{\leq} p \rrbracket_{\bar{e}(V)} \\
 &= (e \circ \iota)(\llbracket p \rrbracket_U) \quad \text{assoc. of } \circ & &= \Box^{\mathbb{B}}(\llbracket p \rrbracket_{\bar{e}(V)}) \\
 &= \Box^{\mathbb{B}}(\llbracket p \rrbracket_U) & &= \Box^{\mathbb{B}}((e \circ V)(p)) \\
 &= \llbracket \Box_{\leq} p \rrbracket_U & &= (e \circ \iota)((e \circ V)(p)) \\
 &= \llbracket \tau(p) \rrbracket_U. & &= e((\iota \circ e)(V(p))) \quad \text{assoc. of } \circ \\
 & & &= e(V(p)) \quad e \circ (\iota \circ e) = e \\
 & & &= e(\llbracket p \rrbracket_V),
 \end{aligned}$$

which proves the base cases of (b) and (a) respectively. As for the inductive step, let  $\varphi := \psi \rightarrow \chi$ . Then, for any  $U \in \mathbb{B}^X$  and  $V \in \mathbb{A}^X$ ,

$$\begin{aligned}
 e(\llbracket \psi \rightarrow \chi \rrbracket_{r(U)}) &= e(\llbracket \psi \rrbracket_{r(U)} \rightarrow^{\mathbb{A}} \llbracket \chi \rrbracket_{r(U)}) \\
 &= e(\iota(\neg^{\mathbb{B}} e(\llbracket \psi \rrbracket_{r(U)}) \vee^{\mathbb{B}} e(\llbracket \chi \rrbracket_{r(U)}))) \quad \text{assumption (5)} \\
 &= e(\iota(\neg^{\mathbb{B}} \llbracket \tau(\psi) \rrbracket_U \vee^{\mathbb{B}} \llbracket \tau(\chi) \rrbracket_U)) \quad \text{(induction hypothesis)} \\
 &= (e \circ \iota)(\neg^{\mathbb{B}} \llbracket \tau(\psi) \rrbracket_U \vee^{\mathbb{B}} \llbracket \tau(\chi) \rrbracket_U) \\
 &= \Box^{\mathbb{B}}(\neg^{\mathbb{B}} \llbracket \tau(\psi) \rrbracket_U \vee^{\mathbb{B}} \llbracket \tau(\chi) \rrbracket_U) \\
 &= \llbracket \Box_{\leq} (\neg \tau(\psi) \vee \tau(\chi)) \rrbracket_U \\
 &= \llbracket \tau(\psi \rightarrow \chi) \rrbracket_U. \\
 \\ 
 e(\llbracket \psi \rightarrow \chi \rrbracket_V) &= e(\llbracket \psi \rrbracket_V \rightarrow^{\mathbb{A}} \llbracket \chi \rrbracket_V) \\
 &= e(\iota(\neg^{\mathbb{B}} e(\llbracket \psi \rrbracket_V) \vee^{\mathbb{B}} e(\llbracket \chi \rrbracket_V))) \quad \text{assumption (5)} \\
 &= e(\iota(\neg^{\mathbb{B}} \llbracket \tau(\psi) \rrbracket_{\bar{e}(V)} \vee^{\mathbb{B}} \llbracket \tau(\chi) \rrbracket_{\bar{e}(V)})) \quad \text{(induction hypothesis)} \\
 &= (e \circ \iota)(\neg^{\mathbb{B}} \llbracket \tau(\psi) \rrbracket_{\bar{e}(V)} \vee^{\mathbb{B}} \llbracket \tau(\chi) \rrbracket_{\bar{e}(V)}) \\
 &= \Box^{\mathbb{B}}(\neg^{\mathbb{B}} \llbracket \tau(\psi) \rrbracket_{\bar{e}(V)} \vee^{\mathbb{B}} \llbracket \tau(\chi) \rrbracket_{\bar{e}(V)}) \\
 &= \llbracket \Box_{\leq} (\tau(\psi) \vee \tau(\chi)) \rrbracket_{\bar{e}(V)} \\
 &= \llbracket \tau(\psi \rightarrow \chi) \rrbracket_{\bar{e}(V)}.
 \end{aligned}$$

The remaining cases are straightforward, and are left to the reader.  $\square$

The following strengthening of Proposition 7 immediately follows from Propositions 9 and 11:

**Corollary 12.** *Let  $\mathbb{A}$  be a Heyting algebra and  $\mathbb{B}$  a Boolean algebra such that  $e : \mathbb{A} \hookrightarrow \mathbb{B}$  and  $\iota : \mathbb{B} \rightarrow \mathbb{A}$  exist as above. Then for all intuitionistic formulas  $\varphi$  and  $\psi$ ,*

$$\mathbb{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbb{B} \models \tau(\varphi) \leq \tau(\psi),$$

where  $\tau$  is the Gödel-Tarski translation.

<sup>5</sup>The assumption that  $e$  is a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$  is needed for the inductive steps relative to  $\perp$ ,  $\top$ ,  $\wedge$ ,  $\vee$  in the proof this proposition, while condition (5) is needed for the step relative to  $\rightarrow$ .

We finish this subsection by showing that every Heyting algebra  $\mathbb{A}$  embeds into a Boolean algebra  $\mathbb{B}$  in the way described at the beginning of the present subsection:

**Proposition 13.** *For every Heyting algebra  $\mathbb{A}$ , there exists a Boolean algebra  $\mathbb{B}$  such that  $\mathbb{A}$  embeds into  $\mathbb{B}$  via some order-embedding  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  which is also a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$  and has a right adjoint  $\iota: \mathbb{B} \rightarrow \mathbb{A}$  verifying condition (5). Finally, these facts lift to the canonical extensions of  $\mathbb{A}$  and  $\mathbb{B}$  as in the following diagram:*

$$\begin{array}{ccc}
 & \overset{i^\pi}{\curvearrowright} & \\
 \mathbb{A}^\delta & \xrightarrow[\quad e^\delta \quad]{\quad \tau \quad} & \mathbb{B}^\delta \\
 \uparrow & & \uparrow \\
 \mathbb{A} & \xrightarrow[\quad e \quad]{\quad \tau \quad} & \mathbb{B} \\
 & \underset{i}{\curvearrowright} &
 \end{array}$$

*Proof.* Via Esakia duality [15], the Heyting algebra  $\mathbb{A}$  can be identified with the algebra of clopen up-sets of its associated Esakia space  $\mathbb{X}_{\mathbb{A}}$ , which is a Priestley space, hence a Stone space. Let  $\mathbb{B}$  be the Boolean algebra of the clopen subsets of  $\mathbb{X}_{\mathbb{A}}$ . Since any clopen up-set is in particular a clopen subset, a natural order embedding  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  exists, which is also a lattice homomorphism between  $\mathbb{A}$  and  $\mathbb{B}$ . This shows the first part of the claim.

As to the second part, notice that Esakia spaces are Priestley spaces in which the downward-closure of a clopen set is a clopen set.

Therefore, we can define the map  $\iota: \mathbb{B} \rightarrow \mathbb{A}$  by the assignment  $b \mapsto \neg((\neg b)\downarrow)$  where  $b$  is identified with its corresponding clopen set in  $\mathbb{X}_{\mathbb{A}}$ ,  $\neg b$  is identified with the relative complement of the clopen set  $b$ , and  $(\neg b)\downarrow$  is defined using the order in  $\mathbb{X}_{\mathbb{A}}$ . It can be readily verified that  $\iota$  is the right adjoint of  $e$  and that moreover condition (5) holds.

Finally,  $e: \mathbb{A} \rightarrow \mathbb{B}$  being also a homomorphism between the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$  implies that  $e$  is smooth and its canonical extension  $e^\delta: \mathbb{A}^\delta \rightarrow \mathbb{B}^\delta$ , besides being an order-embedding, is a complete homomorphism between the lattice reducts of  $\mathbb{A}^\delta$  and  $\mathbb{B}^\delta$  (cf. [18, Corollary 4.8]), and hence is endowed with both a left and a right adjoint. Furthermore, the right adjoint of  $e^\delta$  coincides with  $\iota^\pi$  (cf. [20, Proposition 4.2]). Hence,  $\mathbb{B}^\delta$  can be endowed with a natural structure of S4 bi-modal algebra by defining  $\Box_{\leq}^{\mathbb{B}^\delta}: \mathbb{B}^\delta \rightarrow \mathbb{B}^\delta$  by the assignment  $u \mapsto (e^\delta \circ \iota^\pi)(u)$ , and  $\Diamond_{\geq}^{\mathbb{B}^\delta}: \mathbb{B}^\delta \rightarrow \mathbb{B}^\delta$  by the assignment  $u \mapsto (e^\delta \circ c)(u)$ .  $\square$

## 4 Gödel-Tarski type translations

As discussed in the introduction, only the fragment of the  $\varepsilon$ -Sahlqvist and inductive inequalities of intuitionistic logic for  $\varepsilon$  constantly equal to 1 are translated into Sahlqvist and inductive S4-formulas via Gödel-Tarski translation. Thus, the Gödel-Tarski translation alone is not enough to account for the full Sahlqvist and inductive correspondence theory. In the present section, we look into a family of *Gödel-Tarski type* translations, defined for different languages, to which we apply the template of Section 3.2. The first of them naturally arises by dualizing the setting of Section 3.1

#### 4.1 The co-Gödel-Tarski translation

Fix a denumerable set  $\text{Atprop}$  of propositional variables. The language of co-intuitionistic logic over  $\text{Atprop}$  is given by

$$\mathcal{L}_C \ni \varphi ::= p \mid \perp \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \multimap \varphi.$$

The target language for translating co-intuitionistic logic is that of the normal modal logic  $S4\Diamond$  over  $\text{Atprop}$ , given by

$$\mathcal{L}_{S4\Diamond} \ni \alpha ::= p \mid \perp \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg \alpha \mid \Diamond_{\geq} \alpha.$$

Just like intuitionistic logic, formulas of co-intuitionistic logic can be interpreted on partial orders  $\mathbb{F} = (W, \leq)$  with persistent valuations. Here we only report on the interpretation of  $\Diamond_{\geq}$ -formulas in  $\mathcal{L}_{S4\Diamond}$  and  $\multimap$ -formulas in  $\mathcal{L}_C$ :

$$\begin{array}{ll} \mathbb{F}, w, U \Vdash^* \Diamond_{\geq} \varphi & \text{iff } \mathbb{F}, v, U \Vdash^* \varphi \text{ for some } v \in w \downarrow. \\ \mathbb{F}, w, V \Vdash \varphi \multimap \psi & \text{iff } \mathbb{F}, v, V \nVdash \varphi \text{ and } \mathbb{F}, v, V \Vdash \psi \text{ for some } v \in w \downarrow. \end{array}$$

The language  $\mathcal{L}_C$  is naturally interpreted in co-Heyting algebras. The connective  $\multimap$  is interpreted as the left residual of  $\vee$ . The co-Gödel-Tarski translation is the map  $\sigma: \mathcal{L}_C \rightarrow \mathcal{L}_{S4\Diamond}$  defined by the following recursion:

$$\begin{aligned} \sigma(p) &= \Diamond_{\geq} p \\ \sigma(\perp) &= \perp \\ \sigma(\top) &= \top \\ \sigma(\varphi \wedge \psi) &= \sigma(\varphi) \wedge \sigma(\psi) \\ \sigma(\varphi \vee \psi) &= \sigma(\varphi) \vee \sigma(\psi) \\ \sigma(\varphi \multimap \psi) &= \Diamond_{\geq} (\neg \sigma(\varphi) \wedge \sigma(\psi)) \end{aligned}$$

Next, we show that Proposition 9 applies to the co-Gödel-Tarski translation. We let  $X := \text{AtProp}$ ,  $\mathcal{L}_1 := \mathcal{L}_C$ , and  $\mathcal{L}_2 := \mathcal{L}_{S4\Diamond}$ . Moreover, we let  $\mathbb{A}$  be a co-Heyting algebra, and  $\mathbb{B}$  a Boolean algebra such that an order-embedding  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  exists, which is also a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$ , and has a left adjoint<sup>6</sup>  $c: \mathbb{B} \rightarrow \mathbb{A}$  such that for all  $a, b \in \mathbb{A}$ ,

$$a \multimap^{\mathbb{A}} b = c(\neg^{\mathbb{B}} e(a) \wedge^{\mathbb{B}} e(b)). \quad (6)$$

Then  $\mathbb{B}$  can be endowed with a natural structure of Boolean algebra expansion (BAE) by defining  $\Diamond^{\mathbb{B}}: \mathbb{B} \rightarrow \mathbb{B}$  by the assignment  $b \mapsto (e \circ c)(b)$ . The following is the dual of Proposition 10 and its proof is omitted.

**Proposition 14.** *The BAE  $(\mathbb{B}, \Diamond^{\mathbb{B}})$ , with  $\Diamond^{\mathbb{B}}$  defined above, is normal and is also an  $S4\Diamond$ -modal algebra.*

Finally, we let  $r: \mathbb{B}^X \rightarrow \mathbb{A}^X$  be defined by the assignment  $U \mapsto (c \circ U)$ . The proof of the following proposition is similar to that of Proposition 11, and its proof is omitted.

---

<sup>6</sup>That is,  $c(b) \leq a$  iff  $b \leq e(a)$  for every  $a \in \mathbb{A}$  and  $b \in \mathbb{B}$ . By well known order-theoretic facts (cf. [13]),  $e \circ c$  is an interior operator, that is, for every  $b, b' \in \mathbb{B}$ ,

- c1.  $b \leq (e \circ c)(b)$ ;
- c2. if  $b \leq b'$  then  $(e \circ c)(b) \leq (e \circ c)(b')$ ;
- c3.  $(e \circ c)((e \circ c)(b)) \leq (e \circ c)(b)$ .

Moreover,  $e \circ c \circ e = e$  and  $c = c \circ e \circ c$  (cf. [13, Lemma 7.26]).

**Proposition 15.** *Let  $\mathbb{A}, \mathbb{B}$ ,  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  and  $r: \mathbb{B}^X \rightarrow \mathbb{A}^X$  be as above.<sup>7</sup> Then the co-Gödel-Tarski translation  $\sigma$  satisfies conditions (a) and (b) of Proposition 9 for any formula  $\varphi \in \mathcal{L}_C$ .*

The following corollary immediately follows from Propositions 9 and 15:

**Corollary 16.** *Let  $\mathbb{A}$  be a co-Heyting algebra and  $\mathbb{B}$  a Boolean algebra such that an order-embedding  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  exists, which is a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$ , and has a left adjoint  $c: \mathbb{B} \rightarrow \mathbb{A}$  such that condition (6) holds for all  $a, b \in \mathbb{A}$ . Then for all  $\varphi, \psi \in \mathcal{L}_C$ ,*

$$\mathbb{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbb{B} \models \sigma(\varphi) \leq \sigma(\psi),$$

where  $\sigma$  is the co-Gödel-Tarski translation.

We finish this subsection by showing that every co-Heyting algebra  $\mathbb{A}$  embeds into a Boolean algebra  $\mathbb{B}$  in the way described in Corollary 16:

**Proposition 17.** *For every co-Heyting algebra  $\mathbb{A}$ , there exists a Boolean algebra  $\mathbb{B}$  such that  $\mathbb{A}$  embeds into  $\mathbb{B}$  via some order-embedding  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  which is a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$ , and has a left adjoint  $c: \mathbb{B} \rightarrow \mathbb{A}$  verifying condition (6). Finally, these facts lift to the canonical extensions of  $\mathbb{A}$  and  $\mathbb{B}$  as in the following diagram:*

$$\begin{array}{ccc} \mathbb{A}^\delta & \xrightarrow{e^\delta} & \mathbb{B}^\delta \\ & \begin{array}{c} \xleftarrow{\tau} \\ \xleftarrow{c^\sigma} \end{array} & \\ \uparrow & & \uparrow \\ \mathbb{A} & \xrightarrow{e} & \mathbb{B} \\ & \begin{array}{c} \xleftarrow{\tau} \\ \xleftarrow{c} \end{array} & \end{array}$$

*Proof.* Via Esakia-type duality [15], the co-Heyting algebra  $\mathbb{A}$  can be identified with the algebra of clopen up-sets of its associated co-Esakia space  $\mathbb{X}_{\mathbb{A}}$ , which is a Priestley space, hence a Stone space. Let  $\mathbb{B}$  be the Boolean algebra of the clopen subsets of  $\mathbb{X}_{\mathbb{A}}$ . Since any clopen up-set is in particular a clopen subset, a natural order embedding  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  exists, which is also a lattice homomorphism between  $\mathbb{A}$  and  $\mathbb{B}$ . This shows the first part of the claim.

As to the second part, the duals of Esakia spaces, referred to here as co-Esakia spaces, are Priestley spaces such that the upward-closure of a clopen set is a clopen set.

Therefore, we can define the map  $c: \mathbb{B} \rightarrow \mathbb{A}$  by the assignment  $b \mapsto b\uparrow$  where  $b$  is identified with its corresponding clopen set in  $\mathbb{X}_{\mathbb{A}}$ , where  $b\uparrow$  is defined using the order in  $\mathbb{X}_{\mathbb{A}}$ . It can be readily verified that  $c$  is the left adjoint of  $e$  and that moreover condition (6) holds.

Finally,  $e: \mathbb{A} \rightarrow \mathbb{B}$  being also a homomorphism between the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$  implies that  $e$  is smooth and its canonical extension  $e^\delta: \mathbb{A}^\delta \rightarrow \mathbb{B}^\delta$ , besides being an order-embedding, is a complete homomorphism between the lattice reducts of  $\mathbb{A}^\delta$  and  $\mathbb{B}^\delta$  (cf. [18, Corollary 4.8]), and hence is endowed with both a left and a right adjoint. Furthermore, the left adjoint of  $e^\delta$  coincides with  $c^\sigma$  (cf. [20, Proposition 4.2]). Hence,  $\mathbb{B}^\delta$  can be endowed with a natural structure of S4 bi-modal algebra by defining  $\Box_{\leq}^{\mathbb{B}^\delta}: \mathbb{B}^\delta \rightarrow \mathbb{B}^\delta$  and  $\Diamond_{\geq}^{\mathbb{B}^\delta}: \mathbb{B}^\delta \rightarrow \mathbb{B}^\delta$  by the assignments  $u \mapsto (e^\delta \circ \iota)(u)$ , and  $u \mapsto (e^\delta \circ c^\sigma)(u)$  respectively.  $\square$

<sup>7</sup>The assumption that  $e$  is a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$  is needed for the inductive steps relative to  $\perp, \top, \wedge, \vee$  in the proof this proposition, while condition (6) is needed for the step relative to  $\multimap$ .

## 4.2 Extending the Gödel and co-Gödel translations to bi-intuitionistic logic

The language of bi-intuitionistic logic is given by

$$\mathcal{L}_B \ni \varphi ::= p \mid \perp \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \varphi \multimap \varphi$$

The language of the normal bi-modal logic S4 is given by

$$\mathcal{L}_{S4B} \ni \alpha ::= p \mid \perp \mid \alpha \vee \alpha \mid \neg \alpha \mid \Box_{\leq} \alpha \mid \Diamond_{\geq} \alpha$$

The Gödel-Tarski and the co-Gödel-Tarski translations  $\tau$  and  $\sigma$  can be extended to the bi-intuitionistic language as the maps  $\tau', \sigma' : \mathcal{L}_B \rightarrow \mathcal{L}_{S4B}$  defined by the following recursions:

$$\begin{array}{ll} \tau'(p) &= \Box_{\leq} p & \sigma'(p) &= \Diamond_{\geq} p \\ \tau'(\perp) &= \perp & \sigma'(\perp) &= \perp \\ \tau'(\top) &= \top & \sigma'(\top) &= \top \\ \tau'(\varphi \wedge \psi) &= \tau'(\varphi) \wedge \tau'(\psi) & \sigma'(\varphi \wedge \psi) &= \sigma'(\varphi) \wedge \sigma'(\psi) \\ \tau'(\varphi \vee \psi) &= \tau'(\varphi) \vee \tau'(\psi) & \sigma'(\varphi \vee \psi) &= \sigma'(\varphi) \vee \sigma'(\psi) \\ \tau'(\varphi \rightarrow \psi) &= \Box_{\leq}(\neg \tau'(\varphi) \vee \tau'(\psi)) & \sigma'(\varphi \rightarrow \psi) &= \Box_{\leq}(\neg \sigma'(\varphi) \vee \sigma'(\psi)). \\ \tau'(\varphi \multimap \psi) &= \Diamond_{\geq}(\neg \tau'(\varphi) \wedge \tau'(\psi)) & \sigma'(\varphi \multimap \psi) &= \Diamond_{\geq}(\neg \sigma'(\varphi) \wedge \sigma'(\psi)). \end{array}$$

Notice that  $\tau'$  and  $\sigma'$  agree on each defining clause but those relative to the proposition variables. Let  $\mathbb{A}$  be a bi-Heyting algebra and  $\mathbb{B}$  a Boolean algebra such that  $e : \mathbb{A} \hookrightarrow \mathbb{B}$  is an order-embedding and a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$ . Suppose that  $e$  has both a left adjoint  $c : \mathbb{B} \rightarrow \mathbb{A}$  and a right adjoint  $\iota : \mathbb{B} \rightarrow \mathbb{A}$  such that identities (5) and (6) hold for every  $a, b \in \mathbb{A}$ . Then  $\mathbb{B}$  can be endowed with a natural structure of bi-modal S4-algebra by defining  $\Box^{\mathbb{B}} : \mathbb{B} \rightarrow \mathbb{B}$  by the assignment  $b \mapsto (e \circ \iota)(b)$  and  $\Diamond^{\mathbb{B}} : \mathbb{B} \rightarrow \mathbb{B}$  by the assignment  $b \mapsto (e \circ c)(b)$ .

**Proposition 18.** *The BAE  $(\mathbb{B}, \Box^{\mathbb{B}}, \Diamond^{\mathbb{B}})$ , with  $\Box^{\mathbb{B}}, \Diamond^{\mathbb{B}}$  defined as above, is normal and an S4-bimodal algebra.*

The proof of the following proposition is similar to those of Propositions 11 and 15, and is omitted.

**Proposition 19.** *The translation  $\tau'$  (resp.  $\sigma'$ ) defined above satisfies conditions (a) and (b) of Proposition 9 relative to  $r : \mathbb{B}^X \rightarrow \mathbb{A}^X$  defined by  $U \mapsto (\iota \circ U)$  (resp. defined by  $U \mapsto (c \circ U)$ ).*

Thanks to the proposition above, Proposition 9 applies to both  $\tau'$  and  $\sigma'$ , which provides us with two equally well behaved ways of defining Gödel-Tarski-type translations for the bi-intuitionistic language in a way which retains the main property of the original Gödel-Tarski translation, namely the preservation and reflection of validity over S4-frames. In the light of this result, a natural question is whether  $\tau'$  and  $\sigma'$  are the only two translations with this property. In the following subsection we will answer this question in the negative.

## 4.3 Parametric Gödel-Tarski-type translations for bi-intuitionistic logic

Let  $X := \text{AtProp}$ . For any order-type  $\varepsilon$  on  $X$ , define the translation  $\tau_{\varepsilon} : \mathcal{L}_B \rightarrow \mathcal{L}_{S4B}$  by the following recursion:

$$\tau_{\varepsilon}(p) = \begin{cases} \Box_{\leq} p & \text{if } \varepsilon(p) = 1 \\ \Diamond_{\geq} p & \text{if } \varepsilon(p) = \partial \end{cases}.$$

A similar definition appears in [19]. The remaining defining clauses for  $\tau_\varepsilon$  are analogous to those for  $\tau'$  (see above).<sup>8</sup> Clearly,  $\tau' = \tau_\varepsilon$  for  $\varepsilon$  constantly 1, and  $\sigma' = \tau_\varepsilon$  for  $\varepsilon$  constantly  $\partial$ .

Let  $\mathbb{A}$  be a bi-Heyting algebra and  $\mathbb{B}$  be a Boolean algebra such that an order-embedding  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  exists, which is a homomorphism of the lattice-reducts of  $\mathbb{A}$  and  $\mathbb{B}$ , is endowed with both right and left adjoints, and satisfies (5) and (6) for every  $a, b \in \mathbb{A}$  as described in the previous subsection. For every order-type  $\varepsilon$  on  $X$ , consider the map  $r_\varepsilon: \mathbb{B}^X \rightarrow \mathbb{A}^X$  defined, for any  $U \in \mathbb{B}^X$  and  $p \in X$ , by:

$$r_\varepsilon(U)(p) = \begin{cases} (\iota \circ U)(p) & \text{if } \varepsilon(p) = 1 \\ (c \circ U)(p) & \text{if } \varepsilon(p) = \partial \end{cases}$$

**Proposition 20.** *For every order-type  $\varepsilon$  on  $X$ , the translation  $\tau_\varepsilon$  defined above satisfies conditions (a) and (b) of Proposition 9 relative to  $r_\varepsilon$ .*

*Proof.* By induction on  $\varphi$ . As for the base case, let  $\varphi := p \in \text{AtProp}$ . If  $\varepsilon(p) = \partial$ , then for any  $U \in \mathbb{B}^X$  and  $V \in \mathbb{A}^X$ ,

$$\begin{aligned} e(\llbracket p \rrbracket_{r_\varepsilon(U)}) &= e((c \circ U)(p)) && (\text{def. of } r_\varepsilon) && \llbracket \tau_\varepsilon(p) \rrbracket_{\bar{e}(V)} &= \llbracket \diamond_{\geq} p \rrbracket_{\bar{e}(V)} && (\text{def. of } \tau_\varepsilon) \\ &= (e \circ c)(\llbracket p \rrbracket_U) && (\text{assoc. of } \circ) && &= \diamond^{\mathbb{B}}(\llbracket p \rrbracket_{\bar{e}(V)}) && (\text{def. of } \llbracket \cdot \rrbracket_U) \\ &= \diamond^{\mathbb{B}}(\llbracket p \rrbracket_U) && (\text{def. of } \diamond^{\mathbb{B}}) && &= \diamond^{\mathbb{B}}((e \circ V)(p)) && (\text{def. of } \bar{e}(V)) \\ &= \llbracket \diamond_{\geq} p \rrbracket_U && (\text{def. of } \llbracket \cdot \rrbracket_U) && &= (e \circ c)((e \circ V)(p)) && (\text{def. of } \diamond^{\mathbb{B}}) \\ &= \llbracket \tau_\varepsilon(p) \rrbracket_U. && (\text{def. of } \tau_\varepsilon) && &= e((c \circ e)(V(p))) && (\text{assoc. of } \circ) \\ & & & & &= e(V(p)) && (e \circ (c \circ e) = e) \\ & & & & &= e(\llbracket p \rrbracket_V). && (\text{def. of } \llbracket \cdot \rrbracket_V) \end{aligned}$$

If  $\varepsilon(p) = 1$ , then for any  $U \in \mathbb{B}^X$  and  $V \in \mathbb{A}^X$ ,

$$\begin{aligned} e(\llbracket p \rrbracket_{r_\varepsilon(U)}) &= e((\iota \circ U)(p)) && (\text{def. of } r_\varepsilon) && \llbracket \tau_\varepsilon(p) \rrbracket_{\bar{e}(V)} &= \llbracket \square_{\leq} p \rrbracket_{\bar{e}(V)} && (\text{def. of } \tau_\varepsilon) \\ &= (e \circ \iota)(\llbracket p \rrbracket_U) && (\text{assoc. of } \circ) && &= \square^{\mathbb{B}}(\llbracket p \rrbracket_{\bar{e}(V)}) && (\text{def. of } \llbracket \cdot \rrbracket_U) \\ &= \square^{\mathbb{B}}(\llbracket p \rrbracket_U) && (\text{def. of } \square^{\mathbb{B}}) && &= \square^{\mathbb{B}}((e \circ V)(p)) && (\text{def. of } \bar{e}(V)) \\ &= \llbracket \square_{\leq} p \rrbracket_U && (\text{def. of } \llbracket \cdot \rrbracket_U) && &= (e \circ \iota)((e \circ V)(p)) && (\text{def. of } \square^{\mathbb{B}}) \\ &= \llbracket \tau_\varepsilon(p) \rrbracket_U. && (\text{def. of } \tau_\varepsilon) && &= e((\iota \circ e)(V(p))) && (\text{assoc. of } \circ) \\ & & & & &= e(V(p)) && (e \circ (\iota \circ e) = e) \\ & & & & &= e(\llbracket p \rrbracket_V). && (\text{def. of } \llbracket \cdot \rrbracket_V) \end{aligned}$$

The remainder of the proof is similar to that of Proposition 19 for  $\tau'$ , and is omitted.  $\square$

As a consequence of the proposition above, Proposition 9 applies to  $\tau_\varepsilon$  for any order-type  $\varepsilon$  on  $X$ . Hence:

**Corollary 21.** *Let  $\mathbb{A}$  be a bi-Heyting algebra. If an embedding  $e: \mathbb{A} \rightarrow \mathbb{B}$  exists into a Boolean algebra  $\mathbb{B}$  which is a homomorphism of the lattice reducts and  $e$  has both a right adjoint  $\iota: \mathbb{B} \rightarrow \mathbb{A}$  and a left adjoint  $c: \mathbb{B} \rightarrow \mathbb{A}$  satisfying (5) and (6) for every  $a, b \in \mathbb{A}$ , then for any bi-intuitionistic inequality  $\varphi \leq \psi$ ,*

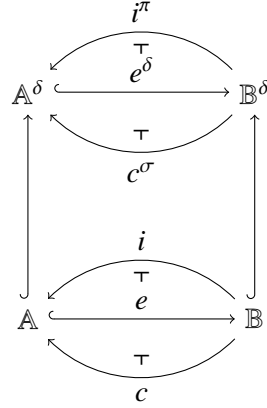
$$\mathbb{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi).$$

We finish this subsection by showing that every bi-Heyting algebra  $\mathbb{A}$  embeds into a Boolean algebra  $\mathbb{B}$  in the way described in Corollary 21:

<sup>8</sup>Dually, we could also define the parametric generalization  $\sigma_\varepsilon$  of  $\sigma$ . Since  $\sigma_\varepsilon = \tau_{\varepsilon^\partial}$ , this definition would be redundant.



**Proposition 22.** *For every bi-Heyting algebra  $\mathbb{A}$ , there exists a Boolean algebra  $\mathbb{B}$  such that  $\mathbb{A}$  embeds into  $\mathbb{B}$  via some order-embedding  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  which is also a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$  and has both a left adjoint  $c: \mathbb{B} \rightarrow \mathbb{A}$  and a right adjoint  $\iota: \mathbb{B} \rightarrow \mathbb{A}$  verifying conditions (5) and (6). Finally, all these facts lift to the canonical extensions of  $\mathbb{A}$  and  $\mathbb{B}$  as in the following diagram:*



*Proof.* Via Esakia-type duality [15], the bi-Heyting algebra  $\mathbb{A}$  can be identified with the algebra of clopen up-sets of its associated dual space  $\mathbb{X}_{\mathbb{A}}$  (referred to here as a bi-Esakia space), which is a Priestley space, hence a Stone space. Let  $\mathbb{B}$  be the Boolean algebra of the clopen subsets of  $\mathbb{X}_{\mathbb{A}}$ . Since any clopen up-set is in particular a clopen subset, a natural order embedding  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  exists, which is also a lattice homomorphism between  $\mathbb{A}$  and  $\mathbb{B}$ . This shows the first part of the claim.

As to the second part, bi-Esakia spaces are Priestley spaces such that both the upward-closure and the downward-closure of a clopen set is a clopen set.

Therefore, we can define the maps  $c: \mathbb{B} \rightarrow \mathbb{A}$  and  $\iota: \mathbb{B} \rightarrow \mathbb{A}$  by the assignments  $b \mapsto b\uparrow$  and  $b \mapsto \neg((-b)\downarrow)$  respectively, where  $b$  is identified with its corresponding clopen set in  $\mathbb{X}_{\mathbb{A}}$ ,  $\neg b$  is defined as the relative complement of  $b$  in  $\mathbb{X}_{\mathbb{A}}$ , and  $b\uparrow$  and  $(\neg b)\downarrow$  are defined using the order in  $\mathbb{X}_{\mathbb{A}}$ . It can be readily verified that  $c$  and  $\iota$  are the left and right adjoints of  $e$  respectively, and that moreover conditions (5) and (6) hold.

Finally,  $e: \mathbb{A} \rightarrow \mathbb{B}$  being also a homomorphism between the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$  implies that  $e$  is smooth and its canonical extension  $e^\delta: \mathbb{A}^\delta \rightarrow \mathbb{B}^\delta$ , besides being an order-embedding, is a complete homomorphism between the lattice reducts of  $\mathbb{A}^\delta$  and  $\mathbb{B}^\delta$  (cf. [18, Corollary 4.8]), and hence is endowed with both a left and a right adjoint. Furthermore, the left (resp. right) adjoint of  $e^\delta$  coincides with  $c^\sigma$  (resp. with  $\iota^\pi$ ) (cf. [20, Proposition 4.2]). Hence,  $\mathbb{B}^\delta$  can be endowed with a natural structure of S4 bi-modal algebra by defining  $\Box_{\leq}^{\mathbb{B}^\delta}: \mathbb{B}^\delta \rightarrow \mathbb{B}^\delta$  by the assignment  $u \mapsto (e^\delta \circ \iota^\pi)(u)$ , and  $\Diamond_{\geq}^{\mathbb{B}^\delta}: \mathbb{B}^\delta \rightarrow \mathbb{B}^\delta$  by the assignment  $u \mapsto (e^\delta \circ c^\sigma)(u)$ .  $\square$

#### 4.4 Parametric Gödel-Tarski-type translations for normal DLEs

Throughout the present section, let us fix a normal DLE-signature  $\mathcal{L}_{\text{DLE}} = \mathcal{L}_{\text{DLE}}(\mathcal{F}, \mathcal{G})$ . The present section is aimed at extending the definition of parametric Gödel-Tarski-type translations from the bi-intuitionistic setting to the general DLE-setting. Towards this aim, we need to define the target language for these translations. This is given in two steps: firstly, we define the normal BAE signature  $\overline{\mathcal{L}}_{\text{BAE}} = \mathcal{L}_{\text{BAE}}(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ , where  $\overline{\mathcal{F}} := \{\overline{f} \mid f \in \mathcal{F}\}$ , and  $\overline{\mathcal{G}} := \{\overline{g} \mid g \in \mathcal{G}\}$ , and for every  $f \in \mathcal{F}$  (resp.  $g \in \mathcal{G}$ ), the connective  $\overline{f}$  (resp.  $\overline{g}$ ) is such that  $n_{\overline{f}} = n_f$  (resp.  $n_{\overline{g}} = n_g$ ) and  $\varepsilon_{\overline{f}}(i) = 1$  for each  $1 \leq i \leq n_f$  (resp.  $\varepsilon_{\overline{g}}(i) = \partial$  for each  $1 \leq i \leq n_g$ ).

Secondly, we assume that an order embedding  $e: \mathbb{A} \hookrightarrow \mathbb{B}$  exists, which is a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$ , is such that both the left and right adjoint  $c: \mathbb{B} \rightarrow \mathbb{A}$  and  $\iota: \mathbb{B} \rightarrow \mathbb{A}$  exist

and moreover the following diagrams commute for every  $f \in \mathcal{F}$  and  $g \in \mathcal{G}$ :<sup>9</sup>

$$\begin{array}{ccc}
\mathbb{A}^{\varepsilon_f} & \xrightarrow{e^{\varepsilon_f}} & \mathbb{B}^{\varepsilon_{\bar{f}}} \\
\downarrow f^{\mathbb{A}} & & \downarrow \bar{f}^{\mathbb{B}} \\
\mathbb{A} & \xleftarrow{c} & \mathbb{B}
\end{array}
\quad
\begin{array}{ccc}
\mathbb{A}^{\varepsilon_g} & \xrightarrow{e^{\varepsilon_g}} & \mathbb{B}^{\varepsilon_{\bar{g}}} \\
\downarrow g^{\mathbb{A}} & & \downarrow \bar{g}^{\mathbb{B}} \\
\mathbb{A} & \xleftarrow{\iota} & \mathbb{B}
\end{array}
\tag{7}$$

Then, as discussed early on, the Boolean reduct of  $\mathbb{B}$  can be endowed with a natural structure of bi-modal S4-algebra by defining  $\Box^{\mathbb{B}} : \mathbb{B} \rightarrow \mathbb{B}$  by the assignment  $b \mapsto (e \circ \iota)(b)$  and  $\Diamond^{\mathbb{B}} : \mathbb{B} \rightarrow \mathbb{B}$  by the assignment  $b \mapsto (e \circ c)(b)$ .

The target language for the parametrized Gödel-Tarski type translations over  $\text{Atprop}$  is given by

$$\mathcal{L}_{BAE}^* \ni \alpha ::= p \mid \perp \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg \alpha \mid \bar{f}(\bar{\alpha}) \mid \bar{g}(\bar{\alpha}) \mid \Diamond_{\geq} \alpha \mid \Box_{\leq} \alpha.$$

Let  $X := \text{AtProp}$ . For any order-type  $\varepsilon$  on  $X$ , define the translation  $\tau_{\varepsilon} : \mathcal{L}_{DLE} \rightarrow \mathcal{L}_{BAE}^*$  by the following recursion:

$$\begin{aligned}
\tau_{\varepsilon}(\perp) &= \perp \\
\tau_{\varepsilon}(\top) &= \top \\
\tau_{\varepsilon}(p) &= \begin{cases} \Box_{\leq} p & \text{if } \varepsilon(p) = 1 \\ \Diamond_{\geq} p & \text{if } \varepsilon(p) = \partial, \end{cases} & \begin{aligned} \tau_{\varepsilon}(\varphi \wedge \psi) &= \tau_{\varepsilon}(\varphi) \wedge \tau_{\varepsilon}(\psi) \\ \tau_{\varepsilon}(\varphi \vee \psi) &= \tau_{\varepsilon}(\varphi) \vee \tau_{\varepsilon}(\psi) \\ \tau_{\varepsilon}(f(\bar{\varphi})) &= \bar{f}(\tau_{\varepsilon}(\bar{\varphi})^{\varepsilon_f}) \\ \tau_{\varepsilon}(g(\bar{\varphi})) &= \bar{g}(\tau_{\varepsilon}(\bar{\varphi})^{\varepsilon_g}) \end{aligned}
\end{aligned}$$

where for each order-type  $\eta$  on  $n$  and any  $n$ -tuple  $\bar{\psi}$  of  $\mathcal{L}_{BAE}$ -formulas,  $\bar{\psi}^{\eta}$  denotes the  $n$ -tuple  $(\psi'_i)_{i=1}^n$ , where  $\psi'_i = \psi_i$  if  $\eta(i) = 1$  and  $\psi'_i = \neg \psi_i$  if  $\eta(i) = \partial$ .

Let  $\mathbb{A}$  be a  $\mathcal{L}_{DLE}$ -algebra and  $\mathbb{B}$  be a  $\mathcal{L}_{BAE}^*$ -algebra such that an order-embedding  $e : \mathbb{A} \hookrightarrow \mathbb{B}$  exists, which is a homomorphism of the lattice-reducts of  $\mathbb{A}$  and  $\mathbb{B}$ , is endowed with both right and left adjoints, and satisfies the commutativity of the diagrams (7) for every  $f \in \mathcal{F}$  and  $g \in \mathcal{G}$ . For every order-type  $\varepsilon$  on  $X$ , consider the map  $r_{\varepsilon} : \mathbb{B}^X \rightarrow \mathbb{A}^X$  defined, for any  $U \in \mathbb{B}^X$  and  $p \in X$ , by:

$$r_{\varepsilon}(U)(p) = \begin{cases} (\iota \circ U)(p) & \text{if } \varepsilon(p) = 1 \\ (c \circ U)(p) & \text{if } \varepsilon(p) = \partial \end{cases}$$

**Proposition 23.** *For every order-type  $\varepsilon$  on  $X$ , the translation  $\tau_{\varepsilon}$  defined above satisfies conditions (a) and (b) of Proposition 9 relative to  $r_{\varepsilon}$ .*

*Proof.* By induction on  $\varphi$ . The base cases are analogous to those in the proof of Proposition 20. Let  $\varphi := f(\bar{\varphi})$ . Then for any  $U \in \mathbb{B}^X$  and  $V \in \mathbb{A}^X$ ,

$$\begin{aligned}
e(\llbracket f(\bar{\varphi}) \rrbracket_{r_{\varepsilon}(U)}) &= e(f(\llbracket \bar{\varphi} \rrbracket_{r_{\varepsilon}(U)})) & (\text{def. of } \llbracket \cdot \rrbracket_{r_{\varepsilon}(U)}) & \quad \llbracket \tau_{\varepsilon}(f(\bar{\varphi})) \rrbracket_{\bar{e}(V)} &= \llbracket \bar{f}(\tau_{\varepsilon}(\bar{\varphi})) \rrbracket_{\bar{e}(V)} & (\text{def. of } \tau_{\varepsilon}) \\
&= \overline{f(e(\llbracket \bar{\varphi} \rrbracket_{r_{\varepsilon}(U)}))} & (\text{assump. (7)}) & \quad &= \overline{f(\llbracket \tau_{\varepsilon}(\bar{\varphi}) \rrbracket_{\bar{e}(V)})} & (\text{def. of } \llbracket \cdot \rrbracket_{\bar{e}(V)}) \\
&= \overline{f(\llbracket \tau_{\varepsilon}(\bar{\varphi}) \rrbracket_U)} & (\text{IH}) & \quad &= \overline{f(e(\llbracket \bar{\varphi} \rrbracket_V))} & (\text{IH}) \\
&= \llbracket \bar{f}(\tau_{\varepsilon}(\bar{\varphi})) \rrbracket_U & (\text{def. of } \llbracket \cdot \rrbracket_U) & \quad &= e(f(\llbracket \bar{\varphi} \rrbracket_V)) & (\text{assump. (7)}) \\
&= \llbracket \tau_{\varepsilon}(f(\bar{\varphi})) \rrbracket_U & (\text{def. of } \tau_{\varepsilon}) & \quad &= e(\llbracket f(\bar{\varphi}) \rrbracket_V) & (\text{def. of } \llbracket \cdot \rrbracket_V)
\end{aligned}$$

For the sake of readability, the polarity bookkeeping  $\bar{\psi}^{\eta}$  (cf. page 18) has been suppressed in the computation above. The remaining cases are analogous and are omitted.  $\square$

<sup>9</sup>Notice that equations (5) and (6) encode the special cases of the commutativity of the diagrams (7) for  $f(\varphi, \psi) := \varphi \succ \psi$  (in which case,  $\bar{f}(\alpha, \beta) := \neg \alpha \wedge \beta$ ) and  $g(\varphi, \psi) := \varphi \rightarrow \psi$  (in which case,  $\bar{g}(\alpha, \beta) := \neg \alpha \vee \beta$ ).

As a consequence of the proposition above, Proposition 9 applies to  $\tau_\varepsilon$  for any order-type  $\varepsilon$  on  $X$ . Hence:

**Corollary 24.** *Let  $\mathbb{A}$  be a  $\mathcal{L}_{\text{DLE}}$ -algebra. If an embedding  $e : \mathbb{A} \hookrightarrow \mathbb{B}$  exists into a  $\mathcal{L}_{\text{BAE}}^*$ -algebra  $\mathbb{B}$  which is a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$ , and  $e$  has both a right adjoint  $\iota : \mathbb{B} \rightarrow \mathbb{A}$  and a left adjoint  $c : \mathbb{B} \rightarrow \mathbb{A}$  satisfying the commutativity of the diagrams (7) for every  $f \in \mathcal{F}$  and  $g \in \mathcal{G}$ , then for any  $\mathcal{L}_{\text{DLE}}$ -inequality  $\varphi \leq \psi$ ,*

$$\mathbb{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi).$$

We finish this subsection by showing that every perfect  $\mathcal{L}_{\text{DLE}}$ -algebra  $\mathbb{A}$  embeds into a perfect Boolean algebra  $\mathbb{B}$  in the way described in Corollary 24:

**Proposition 25.** *For every perfect  $\mathcal{L}_{\text{DLE}}$ -algebra  $\mathbb{A}$ , there exists a perfect  $\mathcal{L}_{\text{BAE}}^*$ -algebra  $\mathbb{B}$  such that  $\mathbb{A}$  embeds into  $\mathbb{B}$  via some order-embedding  $e : \mathbb{A} \hookrightarrow \mathbb{B}$  which is also a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$  and has both a left adjoint  $c : \mathbb{B} \rightarrow \mathbb{A}$  and a right adjoint  $\iota : \mathbb{B} \rightarrow \mathbb{A}$  satisfying the commutativity of the diagrams (7).*

*Proof.* Via Birkhoff duality, the perfect  $\mathcal{L}_{\text{DLE}}$ -algebra  $\mathbb{A}$  can be identified with the algebra of up-sets of its associated prime element structure  $\mathbb{X}_{\mathbb{A}}$ , which is based on a poset. Let  $\mathbb{B}$  be the powerset algebra of the universe of  $\mathbb{X}_{\mathbb{A}}$ . Since any up-set is in particular a subset, a natural order embedding  $e : \mathbb{A} \hookrightarrow \mathbb{B}$  exists, which is also a complete lattice homomorphism between  $\mathbb{A}$  and  $\mathbb{B}$ . This shows the first part of the claim.

As to the second part, notice that the algebras of upsets of a given poset are naturally endowed with a structure of bi-Heyting algebras. Hence we can define the maps  $c : \mathbb{B} \rightarrow \mathbb{A}$  and  $\iota : \mathbb{B} \rightarrow \mathbb{A}$  by the assignments  $b \mapsto b\uparrow$  and  $b \mapsto \neg((\neg b)\downarrow)$  respectively, where  $b$  is identified with its corresponding subset in  $\mathbb{X}_{\mathbb{A}}$ ,  $\neg b$  is defined as the relative complement of  $b$  in  $\mathbb{X}_{\mathbb{A}}$ , and  $b\uparrow$  and  $(\neg b)\downarrow$  are defined using the order in  $\mathbb{X}_{\mathbb{A}}$ . It can be readily verified that  $c$  and  $\iota$  are the left and right adjoints of  $e$  respectively.

Finally, notice that any DLE-frame  $\mathbb{F}$  is also an  $\overline{\mathcal{L}}_{\text{BAE}}^*$ -frame by interpreting the  $f$ -type connective  $\diamond_{\geq}$  by means of the binary relation  $\geq$ , the  $g$ -type connective  $\square_{\leq}$  by means of the binary relation  $\leq$ , each  $\overline{f} \in \overline{\mathcal{F}}$  by means of  $R_f$  and each  $\overline{g} \in \overline{\mathcal{G}}$  by means of  $R_g$ . Moreover, the additional properties (1) and (2) of the relations  $R_f$  and  $R_g$  guarantee that the diagrams (7) commute.  $\square$

Notice that Proposition 25 has a more restricted scope than analogous propositions such as Propositions 22 or 13. Indeed, any DLE  $\mathbb{A}$  is isomorphic via Priestley-type duality to the algebra of clopen up-sets of its dual Priestley space  $\mathbb{X}_{\mathbb{A}}$ , which is a Stone space in particular, and this yields a natural embedding of  $\mathbb{A}$  into the Boolean algebra of the clopen subsets of  $\mathbb{X}_{\mathbb{A}}$ . However, this embedding has in general neither a right nor a left adjoint. In the next section, we will see that Proposition 25 is enough to obtain the Sahlqvist-type correspondence theory for inductive  $\mathcal{L}_{\text{DLE}}$ -inequalities via translation from Sahlqvist-type correspondence theory for inductive  $\mathcal{L}_{\text{BAE}}$ -inequalities. However, we will see in Section 6 that canonicity cannot be straightforwardly obtained in the same way, precisely due to the restriction on Proposition 25.

## 5 Correspondence via translation

The theory developed so far is ready to be applied to the correspondence of inductive DLE-inequalities (hence also intuitionistic, co-intuitionistic and bi-intuitionistic inductive inequalities). In what follows, we let  $\mathcal{L}$  denote any language in  $\{\mathcal{L}_I, \mathcal{L}_C, \mathcal{L}_B, \mathcal{L}_{\text{DLE}}\}$ , and  $\mathcal{L}^*$  its associated target language in  $\{\mathcal{L}_{S4}, \mathcal{L}_{S4\Diamond}, \mathcal{L}_{S4B}, \mathcal{L}_{\text{BAE}}^*\}$ . The general definition of inductive inequalities (cf. Definition 6) applies to each of these languages. In particular, the Boolean negation in any  $\mathcal{L}^*$  enjoys both the order-theoretic

properties of a unary  $f$ -type connective and of a unary  $g$ -type connective. Hence, Boolean negation occurs unrestricted in inductive  $\mathcal{L}^*$ -inequalities. Moreover, the algebraic interpretations of the S4-connectives  $\Box_{\leq}$  and  $\Diamond_{\geq}$  enjoy the order-theoretic properties of normal unary  $f$ -type and  $g$ -type connectives respectively. Hence, the occurrence of  $\Box_{\leq}$  and  $\Diamond_{\geq}$  in inductive  $\mathcal{L}^*$ -inequalities is subject to the same restrictions applied to any connective pertaining to the same class to which they belong.

The following correspondence theorem is a straightforward extension to the  $\mathcal{L}^*$ -setting of the correspondence result for classical normal modal logic in [6]:

**Proposition 26.** *Every inductive  $\mathcal{L}^*$ -inequality has a first-order correspondent over its class of  $\mathcal{L}^*$ -frames.*

In what follows, we aim at obtaining the correspondence theorem for inductive  $\mathcal{L}$ -inequalities from the correspondence theorem for inductive  $\mathcal{L}^*$ -inequalities as stated in the proposition above. Towards this goal, we need the following

**Proposition 27.** *The following are equivalent for any order-type  $\varepsilon$  on  $X$ , and any  $\mathcal{L}$ -inequality  $\varphi \leq \psi$ :*

1.  $\varphi \leq \psi$  is an  $(\Omega, \varepsilon)$ -inductive  $\mathcal{L}$ -inequality;
2.  $\tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi)$  is an  $(\Omega, \varepsilon)$ -inductive  $\mathcal{L}^*$ -inequality.

*Proof.* By induction on the shape of  $\varphi \leq \psi$ . In a nutshell: the definitions involved guarantee that: (1) PIA nodes are introduced immediately above  $\varepsilon$ -critical occurrences of proposition variables; (2) Skeleton nodes are translated as (one or more) Skeleton nodes; (3) PIA nodes are translated as (one or more) PIA nodes. Moreover, this translation does not disturb the dependency order  $\Omega$ . Hence, from item 1 to item 2, the translation does not introduce any violation on  $\varepsilon$ -critical branches, and, from item 2 to item 1, the translation does not amend any violation.  $\square$

**Theorem 5.1** (Correspondence via translation). *Every inductive  $\mathcal{L}$ -inequality has a first-order correspondent on  $\mathcal{L}$ -frames.*

*Proof.* Let  $\varphi \leq \psi$  be an  $(\Omega, \varepsilon)$ -inductive  $\mathcal{L}$ -inequality, and  $\mathbb{F}$  be an  $\mathcal{L}$ -frame such that  $\mathbb{F} \models \varphi \leq \psi$ . By the discrete duality between  $\mathcal{L}$ -algebras and  $\mathcal{L}$ -frames, this assumption is equivalent to  $\mathbb{A} \models \varphi \leq \psi$ , where  $\mathbb{A}$  denotes the complex  $\mathcal{L}$ -algebra of  $\mathbb{F}$ . Since  $\mathbb{A}$  is a perfect  $\mathcal{L}$ -algebra, it is naturally endowed with the structure of a bi-Heyting algebra. By Propositions 13, 17, 22, 25, a perfect  $\mathcal{L}^*$ -algebra  $\mathbb{B}$  exists with a natural embedding  $e : \mathbb{A} \rightarrow \mathbb{B}$  which is a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$  and has both a right adjoint  $\iota : \mathbb{B} \rightarrow \mathbb{A}$  and a left adjoint  $c : \mathbb{B} \rightarrow \mathbb{A}$  such that conditions (5) and (6) hold, and diagrams (7) commute. By Corollaries 21 and 24,  $\mathbb{A} \models \varphi \leq \psi$  iff  $\mathbb{B} \models \tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi)$ , which, by the discrete duality between perfect  $\mathcal{L}^*$ -algebras and  $\mathcal{L}^*$ -frames, is equivalent to  $\mathbb{F} \models^* \tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi)$ .

By Proposition 27,  $\tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi)$  is an  $(\Omega, \varepsilon)$ -inductive  $\mathcal{L}^*$ -inequality, and hence, by Proposition 26,  $\tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi)$  has a first-order correspondent  $\text{FO}(\varphi)$  on  $\mathcal{L}^*$ -frames. Since the first-order theory of  $\mathbb{F}$  as an  $\mathcal{L}$ -frame coincides with the first-order theory of  $\mathbb{F}$  as an  $\mathcal{L}^*$ -frame,  $\text{FO}(\varphi)$  is also the first-order correspondent of  $\varphi \leq \psi$ . The steps of this argument are summarized in the following chain of equivalences:

$$\begin{array}{ll}
\mathbb{F} \models \varphi \leq \psi & \\
\text{iff } \mathbb{A} \models \varphi \leq \psi & \text{(discrete duality for } \mathcal{L}\text{-frames)} \\
\text{iff } \mathbb{B} \models \tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi) & \text{(Corollaries 21 and 24)} \\
\text{iff } \mathbb{F} \models^* \tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi) & \text{(discrete duality for } \mathcal{L}^*\text{-frames)} \\
\text{iff } \mathbb{F} \models \text{FO}(\varphi) & \text{(Proposition 26)}
\end{array}$$

$\square$

**Remark 28.** Theorem 5.1 provides a very concise and uniform route to correspondence for the class of inductive inequalities in any  $\mathcal{L}_{\text{DLE}}$ -signature. This route bypasses one of the two main tools of algorithmic correspondence theory for logics on a weaker than classical propositional base (the algorithm ALBA [8]). Elsewhere [11, 7, 23, 29], evidence was provided to the effect that the scope of applicability of the algorithm ALBA is in fact wider than just the computation of first-order correspondents. Theorem 5.1 shows that in fact, as far as the computation of first-order correspondents is concerned, the algorithm SQEMA, or a suitable generalization of it, is already enough, and ALBA can be actually bypassed when we are only interested in correspondence.

On the other hand, to be able to implement the correspondence-via-translation strategy in a way which is both conceptually significant, and as uniform as in the statement and proof of Theorem 5.1, it is key to employ a definition of Sahlqvist-type formulas or inequalities holding uniformly across signatures, and formulated *independently* of the translations. Such a uniform definition (cf. Definition 6) is the second main tool of unified correspondence theory. Summing up, the translation route to correspondence does not give rise to an alternative ‘unified correspondence theory’ built on independent bases, but is rather facilitated by the notions and insights pertaining unified correspondence theory.

## 6 Canonicity via translation

Recall that  $\mathcal{L}$  denotes any language in  $\{\mathcal{L}_I, \mathcal{L}_C, \mathcal{L}_B, \mathcal{L}_{\text{DLE}}\}$ , and  $\mathcal{L}^*$  its associated target language in  $\{\mathcal{L}_{S4}, \mathcal{L}_{S4\Diamond}, \mathcal{L}_{S4B}, \mathcal{L}_{\text{BAE}}^*\}$ .

The following canonicity theorem is a straightforward reformulation and extension to each  $\mathcal{L}^*$ -setting of the canonicity result for classical normal modal logic in [6]:

**Proposition 29.** *For every inductive  $\mathcal{L}^*$ -inequality  $\alpha \leq \beta$  and every  $\mathcal{L}^*$ -algebra  $\mathbb{B}$ ,*

$$\text{if } \mathbb{B} \models \alpha \leq \beta \text{ then } \mathbb{B}^\delta \models \alpha \leq \beta.$$

In what follows, we aim at obtaining the canonicity theorem for inductive  $\mathcal{L}$ -inequalities from the canonicity theorem for inductive  $\mathcal{L}^*$ -inequalities as stated in the proposition above. While the correspondence-via-translation strategy works uniformly on each  $\mathcal{L}$ -setting, the same is not true for canonicity. In the next subsection we start with the most amenable setting.

### 6.1 Canonicity of inductive inequalities in the bi-intuitionistic setting

In what follows, we aim at obtaining the canonicity theorem for inductive  $\mathcal{L}_B$ -inequalities from the canonicity theorem for inductive  $\mathcal{L}_{S4B}$ -inequalities as stated in Proposition 29.

**Theorem 6.1** (Canonicity via translation). *For every inductive  $\mathcal{L}_B$ -inequality  $\varphi \leq \psi$  and every bi-Heyting algebra  $\mathbb{A}$ ,*

$$\text{if } \mathbb{A} \models \varphi \leq \psi \text{ then } \mathbb{A}^\delta \models \varphi \leq \psi.$$

*Proof.* By Proposition 22, an  $\mathcal{L}_{S4B}$ -algebra  $\mathbb{B}$  exists with a natural embedding  $e : \mathbb{A} \hookrightarrow \mathbb{B}$  which is a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$  and has both a right adjoint  $\iota : \mathbb{B} \rightarrow \mathbb{A}$  and a left adjoint  $c : \mathbb{B} \rightarrow \mathbb{A}$  such that conditions (5) and (6) hold. By Corollary 21,  $\mathbb{A} \models \varphi \leq \psi$  iff  $\mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ .

By Proposition 27,  $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$  is an  $(\Omega, \varepsilon)$ -inductive  $\mathcal{L}_{S4B}$ -inequality, and hence, by Proposition 29,  $\mathbb{B}^\delta \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ . By the last part of the statement of Proposition 22, Corollary 21 applies also to  $\mathbb{A}^\delta$  and  $\mathbb{B}^\delta$ , and thus  $\mathbb{A}^\delta \models \varphi \leq \psi$ , as required. The steps of this argument are summarized in the following U-shaped diagram:

$$\begin{array}{ccc}
\mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\
\Downarrow (\text{Cor 21}) & & \Downarrow (\text{Cor 21}) \\
\mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) & \Leftrightarrow & \mathbb{B}^\delta \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)
\end{array}$$

□

The argument above can be generalized so as to obtain canonicity of inductive inequalities for logics algebraically captured by classes of normal bi-Heyting algebra expansions (cf. Section 2).

## 6.2 Generalizing the canonicity-via-translation argument

In the present subsection, we discuss the extent to which the proof pattern described in the previous subsection can be applied to the settings of Heyting and co-Heyting algebras, and to normal DLEs. In the case of bi-Heyting algebras, the order embedding  $e$ , the existence of which is shown in Proposition 22, has both a left and a right adjoint. This is a major difference with the cases of Heyting and co-Heyting algebras and normal DLEs, in which at most one of the two adjoints exists in general (cf. Propositions 13 and 17), and both adjoints exist if the algebra is perfect.

This implies that the U-shaped argument discussed in the proof of Theorem 6.1, which employed Corollary 21 on both legs as shown in the diagram below, is not available for Heyting/co-Heyting algebras or DLEs. Indeed, in each of these settings, it can still be applied on the side of the perfect algebras, since any such perfect algebra is also a bi-Heyting algebra, but not on general algebras (left-hand side of the diagram).

$$\begin{array}{ccc}
\mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\
\Downarrow ? & & \Downarrow (\text{Cor 21}) \\
\mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) & \Leftrightarrow & \mathbb{B}^\delta \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)
\end{array}$$

In what follows, we aim at giving a refinement of Corollary 21 which can replace the question mark in the U-shaped diagram above. We work in the setting of Heyting algebras (similar statements can be obtained straightforwardly for the other settings as well). Recall that the canonical extension  $e^\delta : \mathbb{A}^\delta \rightarrow \mathbb{B}^\delta$  of the embedding  $e : \mathbb{A} \hookrightarrow \mathbb{B}$  is a complete homomorphism, and hence both its left and right adjoints exist. Let  $c : \mathbb{B}^\delta \rightarrow \mathbb{A}^\delta$  denote the left adjoint of  $e^\delta : \mathbb{A}^\delta \rightarrow \mathbb{B}^\delta$ . Then  $c(b) \in K(\mathbb{A}^\delta)$  for every  $b \in \mathbb{B}$ .<sup>10</sup>

Hence, it is immediate to verify that, if  $r_\varepsilon : (\mathbb{B}^\delta)^X \rightarrow (\mathbb{A}^\delta)^X$  is the map defined for any  $U \in (\mathbb{B}^\delta)^X$  and  $p \in X$  by:

$$r_\varepsilon(U)(p) = \begin{cases} (\iota^\pi \circ U)(p) & \text{if } \varepsilon(p) = 1 \\ (c \circ U)(p) & \text{if } \varepsilon(p) = \partial \end{cases}$$

then,  $(r_\varepsilon(U))(p) \in K(\mathbb{A}^\delta)$  for any ‘admissible valuation’  $U \in \mathbb{B}^X$  and  $p \in X$ . Moreover, since the connective  $\succ$  is not part of the intuitionistic language considered here, and since, as discussed in the proof of Proposition 13, condition (5) lifts from  $e$  and  $\iota$  to  $e^\delta$  and  $\iota^\pi$ , this is enough to show, by induction on the complexity of  $\mathcal{L}_I$ -formulas, that conditions (a) and (b) of Proposition 9 hold relative to  $\tau_\varepsilon$  and  $r_\varepsilon$  defined above.

The following proposition is the required refinement of Corollary 21 which can replace the question mark in the U-shaped diagram above.

<sup>10</sup>Indeed,  $e^\delta$ , being a complete homomorphism, is in particular a box-type map, of which its left adjoint  $c$  is then the ‘black-diamond’ (in the notation of [9]), and it is well-known from the theory of canonical extensions of box-type operators that their left adjoints send closed elements to closed elements.

**Proposition 30.** *Let  $\mathbb{A}$  be a Heyting algebra, and  $e : \mathbb{A} \hookrightarrow \mathbb{B}$  be an embedding of  $\mathbb{A}$  into a Boolean algebra  $\mathbb{B}$  which is a homomorphism of the lattice reducts of  $\mathbb{A}$  and  $\mathbb{B}$ , endowed with its right adjoint  $\iota : \mathbb{B} \rightarrow \mathbb{A}$  so that condition (5) holds. Then, for every  $(\Omega, \varepsilon)$ -inductive  $\mathcal{L}_I$ -inequality  $\varphi \leq \psi$ ,*

$$\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi \quad \text{iff} \quad \mathbb{B}^\delta \models_{\mathbb{B}} \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi).$$

*Sketch of proof.* From right to left, if  $(\mathbb{A}^\delta, V) \not\models \varphi \leq \psi$  for some  $V \in \mathbb{A}^X$ , then  $\llbracket \varphi \rrbracket_V \not\leq \llbracket \psi \rrbracket_V$ . Since  $e : \mathbb{A} \hookrightarrow \mathbb{B}$  is an order-embedding, and as discussed above, conditions (a) and (b) of Proposition 9 hold relative to  $\tau_\varepsilon$  and  $r_\varepsilon : (\mathbb{B}^\delta)^X \rightarrow (\mathbb{A}^\delta)^X$ , this implies that  $\llbracket \tau_\varepsilon(\varphi) \rrbracket_{\bar{e}(V)} = e(\llbracket \varphi \rrbracket_V) \not\leq e(\llbracket \psi \rrbracket_V) = \llbracket \tau_\varepsilon(\psi) \rrbracket_{\bar{e}(V)}$ , that is  $(\mathbb{B}^\delta, \bar{e}(V)) \not\models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ , as required.

Conversely, assume contrapositively that  $(\mathbb{B}^\delta, U) \not\models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$  for some  $U \in \mathbb{B}^X$ , that is,  $\llbracket \tau_\varepsilon(\varphi) \rrbracket_U \not\leq \llbracket \tau_\varepsilon(\psi) \rrbracket_U$ . By applying condition (b) of Proposition 9, this is equivalent to  $e(\llbracket \varphi \rrbracket_{r_\varepsilon(U)}) \not\leq e(\llbracket \psi \rrbracket_{r_\varepsilon(U)})$ , which, by the monotonicity of  $e$ , implies that  $\llbracket \varphi \rrbracket_{r_\varepsilon(U)} \not\leq \llbracket \psi \rrbracket_{r_\varepsilon(U)}$ , that is,  $(\mathbb{A}, r_\varepsilon(U)) \not\models \varphi \leq \psi$ . This is not enough to finish the proof, since  $r_\varepsilon(U)$  is not guaranteed to belong in  $\mathbb{A}^X$ ; however, as observed above,  $r_\varepsilon(U)(p) \in K(\mathbb{A}^\delta)$  for each proposition variable  $p$ . To finish the proof, we need to show that an admissible valuation  $V' \in \mathbb{A}^X$  can be manufactured from  $r_\varepsilon(U)$  and  $\varphi \leq \psi$  in such a way that  $(\mathbb{A}^\delta, V') \not\models \varphi \leq \psi$ . In what follows, we provide a sketch of the proof of the existence of the required  $V'$ . Clearly, if  $\varepsilon(p) = 1$  for every proposition variable  $p$  occurring in  $\varphi \leq \psi$ , then  $r_\varepsilon(U)(p) = \iota^\pi(\llbracket p \rrbracket_U) = \iota(\llbracket p \rrbracket_U) \in \mathbb{A}$ , and then any  $V' \in \mathbb{A}^X$  which agrees with  $r_\varepsilon(U)$  on all variables occurring in  $\varphi \leq \psi$  would be enough to finish the proof. Assume that  $\varepsilon(q) = \partial$  for some proposition variable  $q$  occurring in  $\varphi \leq \psi$ . Then we define  $V'(q) \in \mathbb{A}$  as follows. We run ALBA on  $\varphi \leq \psi$  according to the solving order  $\Omega$ , up to the point when we solve for the negative occurrences of  $q$ , which by assumption are  $\varepsilon$ -critical. Notice that ALBA preserves truth under assignments.<sup>11</sup> Then the inequality providing the minimal valuation of  $q$  is of the form  $q \leq \alpha$ , where  $\alpha$  is *pure* (i.e. no proposition variables occur in  $\alpha$ ). By Lemma 9.5 in [8], every inequality in the antecedent of the quasi-inequality obtained by applying first approximation to an inductive inequality is of the form  $\gamma \leq \delta$  with  $\gamma$  syntactically closed and  $\delta$  syntactically open. Hence,  $\alpha$  is pure and syntactically open, which means that the interpretation of  $\alpha$  is an element in  $O(\mathbb{A}^\delta)$ . Therefore, by compactness, there exists some  $a \in \mathbb{A}$  such that  $r_\varepsilon(U)(q) \leq a \leq \alpha$ . Then we define  $V'(q) = a$ . Finally, it remains to be shown that  $(\mathbb{A}^\delta, V') \not\models \varphi \leq \psi$ . This immediately follows from the fact that ALBA steps preserves truth under assignments, and that all the inequalities in the system are preserved in the change from  $r_\varepsilon(U)$  to  $V'$ .  $\square$

However, having replaced Corollary 21 with Proposition 30 is still not enough for the U-shaped argument above to go through. Indeed, notice that, if  $\varphi \leq \psi$  contains some  $q$  with  $\varepsilon(q) = \partial$ , then  $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$  contains occurrences of the connective  $\diamond_{\geq}$ , the algebraic interpretation of which in  $\mathbb{B}^\delta$  is based on the left adjoint  $c$  of  $e^\delta$ , which, as discussed above, maps elements in  $\mathbb{B}$  to elements in  $K(\mathbb{B}^\delta)$ . Hence, the canonicity of  $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ , understood as the preservation of its validity from  $\mathbb{B}$  to  $\mathbb{B}^\delta$ , cannot be argued by appealing to Proposition 29: indeed, Proposition 29 holds under the assumption that  $\mathbb{B}$  is an  $\mathcal{L}^*$ -subalgebra of  $\mathbb{B}^\delta$ , while, as discussed above,  $\mathbb{B}$  is not in general closed under  $\diamond_{\geq}$ .

In order to be able to adapt the canonicity-via-translation argument to the case of Heyting algebras (or co-Heyting algebras, or normal DLEs), we would need to strengthen Proposition 29 so as to obtain the following equivalence for any inductive  $\mathcal{L}^*$ -inequality  $\alpha \leq \beta$ :

$$\mathbb{B}^\delta \models_{\mathbb{B}} \alpha \leq \beta \quad \text{iff} \quad \mathbb{B}^\delta \models \alpha \leq \beta \tag{8}$$

<sup>11</sup>In [8] it is proved that ALBA steps preserve validity of quasi-inequalities. In fact, it ensures something stronger, namely that truth under assignments is preserved, modulo the values of introduced and eliminated variables. This notion of equivalence is studied in e.g. [5]. We are therefore justified in our assumption that the value of  $q$  is held constant as are the values of all variables occurring in  $\varphi \leq \psi$  which have not yet been eliminated up to the point where  $q$  is solved for.

in a setting in which the subalgebra  $\mathbb{B}$  is not required to be an  $\mathcal{L}^*$ -subalgebra of  $\mathbb{B}^\delta$ , and  $f(\bar{b}) \in K(\mathbb{B}^\delta)$  for every  $f$ -type connective in  $\mathcal{L}^*$  and  $\bar{b} \in \mathbb{B}^{n_f}$ , and  $g(\bar{b}) \in O(\mathbb{B}^\delta)$  for every  $g$ -type connective in  $\mathcal{L}^*$  and  $\bar{b} \in \mathbb{B}^{n_g}$ .

Such a strengthening cannot be straightforwardly obtained with the tools provided by the present state-of-the-art in canonicity theory. To see where the problem lies, let us try and apply ALBA/SQEMA in an attempt to prove the left-to-right direction of (8) for the ‘Sahlqvist’ inequality  $\Box_{\leq} p \leq \Diamond_{\geq} \Box_{\leq} p$ , assuming that  $\Diamond_{\geq}$  is left adjoint to  $\Box_{\leq}$ , and  $\llbracket \Box_{\leq} p \rrbracket_U \in O(\mathbb{B}^\delta)$  and  $\llbracket \Diamond_{\geq} p \rrbracket_U \in K(\mathbb{B}^\delta)$  for any admissible valuation  $U \in \mathbb{B}^X$ :

$$\begin{aligned} & \mathbb{B}^\delta \models_{\mathbb{B}} \forall p [\Box_{\leq} p \leq \Diamond_{\geq} \Box_{\leq} p] \\ \text{iff } & \mathbb{B}^\delta \models_{\mathbb{B}} \forall p \forall \mathbf{i} \forall \mathbf{m} [(\mathbf{i} \leq \Box_{\leq} p \ \& \ \Diamond_{\geq} \Box_{\leq} p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ \text{iff } & \mathbb{B}^\delta \models_{\mathbb{B}} \forall p \forall \mathbf{i} \forall \mathbf{m} [(\Diamond_{\geq} \mathbf{i} \leq p \ \& \ \Diamond_{\geq} \Box_{\leq} p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \end{aligned}$$

The minimal valuation term  $\Diamond_{\geq} \mathbf{j}$ , computed by ALBA/SQEMA when solving for the negative occurrence of  $p$ , is closed. However, substituting this minimal valuation into  $\Diamond_{\geq} \Box_{\leq} p \leq \mathbf{m}$  would get us  $\Diamond_{\geq} \Box_{\leq} \Diamond_{\geq} \mathbf{j} \leq \mathbf{m}$  with  $\Diamond_{\geq} \Box_{\leq} \Diamond_{\geq} \mathbf{j}$  neither closed nor open. Hence, we cannot anymore appeal to the Esakia lemma in order to prove the following equivalence:<sup>12</sup>

$$\begin{aligned} & \mathbb{B}^\delta \models_{\mathbb{B}} \forall p \forall \mathbf{i} \forall \mathbf{m} [(\Diamond_{\geq} \mathbf{i} \leq p \ \& \ \Diamond_{\geq} \Box_{\leq} p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ \text{iff } & \mathbb{B}^\delta \models_{\mathbb{B}} \forall \mathbf{i} \forall \mathbf{m} [\Diamond_{\geq} \Box_{\leq} \Diamond_{\geq} \mathbf{i} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \end{aligned}$$

An analogous situation arises when solving for the positive occurrence of  $p$ . Other techniques for proving canonicity, such as Jónsson-style canonicity [24, 29], display the same problem, since they also rely on an Esakia lemma which is not available if  $\mathbb{B}$  is not closed under  $\Box_{\leq}$  and  $\Diamond_{\geq}$ .

## 7 Conclusions

From the results of this paper, bi-intuitionistic logic stands out as a particularly well behaved setting, and its performance compares favourably to that of the better known intuitionistic logic. Another advantage of bi-intuitionistic logic over intuitionistic logic is that for each additional  $\Box$ -type connective it possible to define a dual *normal* diamond along the usual Boolean pattern as  $\Diamond \varphi := \top \multimap \Box(\varphi \rightarrow \perp)$ , and likewise for each additional  $\Diamond$ -type connective a dual *normal* box as  $\Box \varphi := \Diamond(\top \multimap \varphi) \rightarrow \perp$ . Following this pattern in the intuitionistic case, using only the intuitionistic negation, gives rise to connectives which are monotone but neither regular nor normal. Together with the fact that bi-intuitionistic logic is sound and complete w.r.t. partial orders, this makes bi-intuitionistic logic a particularly attractive basic framework.

We saw that, where applicable, the translation method has an extraordinary synthesizing power. However, as already mentioned at various points early on, we do not believe that the translation approach can provide autonomous foundations to correspondence theory for nonclassical logics, and this for two reasons. First, in Section 6 we saw that, outside of the bi-intuitionistic setting, it is not clear how the canonicity-via-translation argument could be made to work. To make it work, one would likely need techniques which import novel order-theoretic and topological insights which go well beyond the scope of the translation method itself. The existing canonicity techniques therefore remain the most straightforward route toward the result. Second, as argued in Remark 28, unified correspondence is also needed to provide the right background framework in which correspondence-via-translation can be meaningfully investigated.

<sup>12</sup>In other words, if  $\mathbb{B}$  is not closed under  $\Diamond_{\geq}$  or  $\Box_{\leq}$ , the soundness of the application of the Ackermann rule under admissible assignments cannot be argued anymore by appealing to the Esakia lemma, and hence, to the topological Ackermann lemma.



The progress we made over [19] (namely the canonicity-via-translation for the bi-intuitionistic setting) comes from embracing the full extent of the algebraic analysis. Specifically, canonicity-via-translation hinges upon the fact that the interplay of persistent and non-persistent valuations on frames can be understood and reformulated in terms of an adjunction situation between two complex algebras of the same frame. In its turn, this adjunction situation generalizes to arbitrary algebras. The same *modus operandi*, which achieves generalization through algebras via duality, has been fruitfully employed by some of the authors also for very different purposes, such as the definition of the non-classical counterpart of a given logical framework (cf. [26, 3]).

## References

- [1] W. Conradie and A. Craig. Canonicity results for mu-calculi: An algorithmic approach. *Journal of Logic and Computation*, forthcoming.
- [2] W. Conradie, Y. Fomatati, A. Palmigiano, and S. Sourabh. Algorithmic correspondence for intuitionistic modal mu-calculus. *Theoretical Computer Science*, 564:30–62, 2015.
- [3] W. Conradie, S. Frittella, A. Palmigiano, and A. Tzimoulis. Probabilistic epistemic updates on algebras. In *Logic, Rationality, and Interaction*, pages 64–76. Springer, 2015.
- [4] W. Conradie, S. Ghilardi, and A. Palmigiano. Unified correspondence. In A. Baltag and S. Smets, editors, *Johan van Benthem on Logic and Information Dynamics*, volume 5 of *Outstanding Contributions to Logic*, pages 933–975. Springer International Publishing, 2014.
- [5] W. Conradie and V. Goranko. Algorithmic correspondence and completeness in modal logic IV: Semantic extensions of SQEMA. *Journal of Applied Non-Classical Logics*, 18(2-3):175–211, 2008.
- [6] W. Conradie, V. Goranko, and D. Vakarelov. Algorithmic correspondence and completeness in modal logic. I. The core algorithm SQEMA. *Logical Methods in Computer Science*, 2006.
- [7] W. Conradie and A. Palmigiano. Constructive canonicity of inductive inequalities. Submitted.
- [8] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for distributive modal logic. *Annals of Pure and Applied Logic*, 163(3):338 – 376, 2012.
- [9] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for non-distributive logics. *Journal of Logic and Computation*, forthcoming.
- [10] W. Conradie, A. Palmigiano, and S. Sourabh. Algorithmic modal correspondence: Sahlqvist and beyond. Submitted.
- [11] W. Conradie, A. Palmigiano, S. Sourabh, and Z. Zhao. Canonicity and relativized canonicity via pseudo-correspondence: an application of ALBA. Submitted.
- [12] W. Conradie and C. Robinson. On Sahlqvist theory for hybrid logics. *Journal of Logic and Computation*, 2015.
- [13] B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 2002.
- [14] M. de Rijke and Y. Venema. Sahlqvist’s theorem for Boolean algebras with operators with an application to cylindric algebras. *Studia Logica*, 54(1):61–78, 1995.

- [15] L. Esakia. The problem of dualism in the intuitionistic logic and Brouwerian lattices. *The Fifth International Congress of Logic, Methodology and Philosophy of Science*, pages 7–8, 1975.
- [16] S. Frittella, A. Palmigiano, and L. Santocanale. Dual characterizations for finite lattices via correspondence theory for monotone modal logic. *Journal of Logic and Computation*, forthcoming.
- [17] M. Gehrke. Generalized Kripke frames. *Studia Logica*, 84(2):241–275, 2006.
- [18] M. Gehrke and J. Harding. Bounded lattice expansions. *Journal of Algebra*, 238(1):345–371, 2001.
- [19] M. Gehrke, H. Nagahashi, and Y. Venema. A Sahlqvist theorem for distributive modal logic. *Annals of Pure and Applied Logic*, 131(1-3):65–102, 2005.
- [20] M. Gehrke and H. Priestley. Canonical extensions of double quasioperator algebras: an algebraic perspective on duality for certain algebras with binary operations. *Journal of Pure and Applied Algebra*, 209(1):269–290, 2007.
- [21] K. Gödel. Eine interpretation des intuitionistischen aussagenkalküls. *Ergebnisse eines mathematischen Kolloquiums*, 6:39–40, 1933.
- [22] V. Goranko and D. Vakarelov. Elementary canonical formulae: Extending Sahlqvist’s theorem. *Annals of Pure and Applied Logic*, 141(1-2):180–217, 2006.
- [23] G. Greco, M. Ma, A. Palmigiano, A. Tzimoulis, and Z. Zhao. Unified correspondence as a proof-theoretic tool. *Journal of Logic and Computation*, forthcoming.
- [24] B. Jónsson. On the canonicity of Sahlqvist identities. *Studia Logica*, 53:473–491, 1994.
- [25] N. Kurtonina. Categorical inference and modal logic. *Journal of Logic, Language, and Information*, 7, 1998.
- [26] M. Ma, A. Palmigiano, and M. Sadrzadeh. Algebraic semantics and model completeness for intuitionistic public announcement logic. *Annals of Pure and Applied Logic*, 165(4):963–995, 2014.
- [27] J. C. C. McKinsey and A. Tarski. Some theorems about the sentential calculi of Lewis and Heyting. *The Journal of Symbolic Logic*, 13(1):1–15, 1948.
- [28] H. J. Ohlbach and R. A. Schmidt. Functional translation and second-order frame properties of modal logics. *Journal of Logic and Computation*, 7(5):581–603, 1997.
- [29] A. Palmigiano, S. Sourabh, and Z. Zhao. Jónsson-style canonicity for ALBA-inequalities. *Journal of Logic and Computation*, 2015.
- [30] A. Palmigiano, S. Sourabh, and Z. Zhao. Sahlqvist theory for impossible worlds. *Journal of Logic and Computation*, forthcoming.
- [31] H. Sahlqvist. Completeness and correspondence in the first and second order semantics for modal logic. In S. Kanger, editor, *Studies in Logic and the Foundations of Mathematics*, volume 82, pages 110–143. North-Holland, Amsterdam, 1975.
- [32] G. Sambin and V. Vaccaro. A new proof of Sahlqvist’s theorem on modal definability and completeness. *Journal of Symbolic Logic*, 54(3):992–999, 1989.

- [33] B. ten Cate, M. Marx, and J. P. Viana. Hybrid logics with Sahlqvist axioms. *Logic Journal of the IGPL*, (3):293–300, 2006.
- [34] J. van Benthem. Modal reduction principles. *Journal of Symbolic Logic*, 41(2):301–312, 06 1976.
- [35] J. van Benthem. Modal frame correspondences and fixed-points. *Studia Logica*, 83(1-3):133–155, 2006.
- [36] J. van Benthem, N. Bezhanishvili, and I. Hodkinson. Sahlqvist correspondence for modal mu-calculus. *Studia Logica*, 100(1-2):31–60, 2012.
- [37] J. van Benthem, N. Bezhanishvili, and W. Holliday. A bimodal perspective on possibility semantics. ILLC Publications, Prepublication (PP) Series PP-2016-04, University of Amsterdam, 2016.
- [38] F. Wolter and M. Zakharyashev. On the relation between intuitionistic and classical modal logics. *Algebra and Logic*, 36:121–125, 1997.
- [39] F. Wolter and M. Zakharyashev. Intuitionistic modal logics as fragments of classical modal logics. In E. Orłowska, editor, *Logic at Work, Essays in honour of Helena Rasiowa*, pages 168–186. Springer–Verlag, 1998.